

MAYER-VIETORIS SEQUENCE FOR DIFFERENTIABLE/DIFFEOLOGICAL SPACES

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ABSTRACT. The idea of a space with smooth structure is a generalization of an idea of a manifold. K. T. Chen introduced such a space as a differentiable space in his study of a loop space to employ the idea of iterated path integrals [2, 3, 4, 5]. Following the pattern established by Chen, J. M. Souriau [10] introduced his version of a space with smooth structure, which is called a diffeological space. These notions are strong enough to include all the topological spaces. However, if one tries to show de Rham theorem, he must encounter a difficulty to obtain a partition of unity and thus the Mayer-Vietoris exact sequence in general. In this paper, we introduce a new version of differential forms to obtain a partition of unity, the Mayer-Vietoris exact sequence and a version of de Rham theorem in general. In addition, if we restrict ourselves to consider only CW complexes, we obtain de Rham theorem for a genuine de Rham complex, and hence the genuine de Rham cohomology coincides with the ordinary cohomology for a CW complex.

In this paper, we deal with both differentiable and diffeological spaces. A differentiable space is introduced by K. T. Chen [5] and a diffeological space is introduced by J. M. Souriau [10]. Both of them are developed with an idea of a *plot* – a map from a *domain*.

Let $n \geq 0$. A non-void open set in \mathbb{R}^n is called an *open n -domain* or simply an *open domain* and a convex set with non-void interior in \mathbb{R}^n is called a *convex n -domain* or simply a *convex domain*. We reserve the word ‘smooth’ for ‘differentiable infinitely many times’ in the ordinary sense. More precisely, a map from an open or convex domain A to an euclidean space is smooth on A , if it is smooth on $\text{Int } A$ in the ordinary sense and all derivatives extend continuously and uniquely to A (see A. Kriegel and P. W. Michor [9]).

1. DIFFERENTIABLE/DIFFEOLOGICAL SPACES

Let us recall a concrete site given by Chen [5] (see J. C. Baes and A. E. Hoffnung [1]).

Definition 1.1. *Let \mathbf{Convex} be the category of convex domains and smooth maps between them. Then \mathbf{Convex} is a concrete site with Chen’s coverage: a covering family on a convex domain is an open covering by convex domains.*

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On the other hand, a concrete site given by Souriau [10] is as follows.

Definition 1.2. Let \mathbf{Open} be the category of open domains and smooth maps between them. Then \mathbf{Open} is a concrete site with the usual coverage: a covering family on an open domain is an open covering by open domains.

Let \mathbf{Set} be the category of sets. A differentiable or diffeological space is as follows.

Definition 1.3 (Differentiable space). A differentiable space is a pair (X, \mathcal{C}^X) of a set X and a contravariant functor $\mathcal{C}^X : \mathbf{Convex} \rightarrow \mathbf{Set}$ such that

- (C0) For any $A \in \mathcal{O}(\mathbf{Convex})$, $\mathcal{C}^X(A) \subset \mathcal{M}(\mathbf{Set})(A, X)$.
- (C1) For any $x \in X$ and any $A \in \mathcal{O}(\mathbf{Convex})$, $\mathcal{C}^X(A) \ni c_x$ the constant map.
- (C2) Let $A \in \mathcal{O}(\mathbf{Convex})$ with an open covering $A = \bigcup_{\alpha \in \Lambda} \text{Int}_A B_\alpha$, $B_\alpha \in \mathcal{O}(\mathbf{Convex})$. If $P \in \mathcal{M}(\mathbf{Set})(A, X)$ satisfies that $P|_{B_\alpha} \in \mathcal{C}^X(B_\alpha)$ for all $\alpha \in \Lambda$, then $P \in \mathcal{C}^X(A)$.
- (C3) For any $A, B \in \mathcal{O}(\mathbf{Convex})$ and any $f \in \mathcal{M}(\mathbf{Convex})(B, A)$, $\mathcal{C}^X(f) = f^* : \mathcal{C}^X(A) \rightarrow \mathcal{C}^X(B)$ is given by $f^*(P) = P \circ f \in \mathcal{C}^X(A)$ for any $P \in \mathcal{C}^X(A)$.

Definition 1.4 (Diffeological space). A diffeological space is a pair (X, \mathcal{D}^X) of a set X and a contravariant functor $\mathcal{D}^X : \mathbf{Open} \rightarrow \mathbf{Set}$ such that

- (D0) For any $U \in \mathcal{O}(\mathbf{Open})$, $\mathcal{D}^X(U) \subset \text{Map}(U, X)$.
- (D1) For any $x \in X$ and any $U \in \mathcal{O}(\mathbf{Open})$, $\mathcal{D}^X(U) \ni c_x$ the constant map.
- (D2) Let $U \in \mathcal{O}(\mathbf{Open})$ with an open covering $U = \bigcup_{\alpha \in \Lambda} V_\alpha$, $V_\alpha \in \mathcal{O}(\mathbf{Open})$. If $P \in \mathcal{M}(\mathbf{Set})(U, X)$ satisfies that $P|_{V_\alpha} \in \mathcal{D}^X(V_\alpha)$ for all $\alpha \in \Lambda$, then $P \in \mathcal{D}^X(U)$.
- (D3) For any $U, V \in \mathcal{O}(\mathbf{Open})$ and any $f \in \mathcal{M}(\mathbf{Open})(V, U)$, $\mathcal{D}^X(f) = f^* : \mathcal{D}^X(V) \rightarrow \mathcal{D}^X(U)$ is given by $f^*(P) = P \circ f \in \mathcal{D}^X(V)$ for any $P \in \mathcal{D}^X(U)$.

From now on, $\mathcal{E}^X : \mathbf{Domain} \rightarrow \mathbf{Set}$ stands for either $\mathcal{C}^X : \mathbf{Convex} \rightarrow \mathbf{Set}$ or $\mathcal{D}^X : \mathbf{Open} \rightarrow \mathbf{Set}$ to discuss about a differentiable space and a diffeological space simultaneously.

Definition 1.5. A subset $O \subset X$ is open if, for any $P \in \mathcal{E}^X$ ($\mathcal{E} = \mathcal{C}$ or \mathcal{D}), $P^{-1}(O)$ is open in $\text{Dom } P$. When any compact subset of X is closed, we say X is ‘weakly-separated’.

Definition 1.6. Let (X, \mathcal{E}^X) and (Y, \mathcal{E}^Y) be differentiable/diffeological spaces, $\mathcal{E} = \mathcal{C}$ or \mathcal{D} . A map $f : X \rightarrow Y$ is differentiable, if there exists a natural transformation of contravariant functors $\mathcal{E}^f : \mathcal{E}^Y \rightarrow \mathcal{E}^X$ such that $\mathcal{E}^f(P) = f \circ P$. The set of differentiable maps between X and Y is denoted by $C_{\mathcal{E}}^\infty(X, Y)$ or simply by $C^\infty(X, Y)$. If further, f is invertible with a differentiable inverse map, f is said to be a diffeomorphism.

Let us summarize the minimum notions from [2, 3, 4, 5, 10, 1, 12, 7, 6, 8] to build up de Rham theory in the category of differentiable or diffeological spaces as follows.

Definition 1.7 (External algebra). Let $T_n^* = \text{Hom}(\mathbb{R}^n, \mathbb{R}) = \bigoplus_{i=1}^n \mathbb{R} dx_i$, where $\{dx_i\}_{1 \leq i \leq n}$ is the dual basis to the standard basis $\{e_i\}_{1 \leq i \leq n}$ of \mathbb{R}^n . We denote by $\wedge^*(T_n^*)$ the exterior (graded) algebra on $\{dx_i\}$, where each dx_i is of dimension 1. In particular, we have $\wedge^0(T_n^*) \cong \wedge^*(T_0^*) \cong \mathbb{R}$, $\wedge^p(T_n^*) = 0$ if $p < 0$ and $\wedge^p(T_n^*) \cong \wedge^{n-p}(T_n^*)$ for any $p \in \mathbb{Z}$.

The external algebra fits in with our categorical context as the following form.

Definition 1.8. A contravariant functor $\wedge^p : \text{Domain} \rightarrow \text{Set}$ is given as follows:

- (1) $\wedge^p(A) = \mathcal{M}(\text{Domain})(A, \wedge^p(T_n^*))$, for any convex n -domain A ,
- (2) For a smooth map $f : B \rightarrow A$ in **Domain**, $\wedge^p(f) = f^* : \wedge^p(A) \rightarrow \wedge^p(B)$ is defined, for any $\omega = \sum_{i_1 < \dots < i_p} a_{i_1, \dots, i_p}(\mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p} \in \wedge^p(A)$, as

$$f^*(\omega) = \sum_{j_1 < \dots < j_p} b_{j_1, \dots, j_p}(\mathbf{y}) \cdot dy_{j_1} \wedge \dots \wedge dy_{j_p}, \quad \mathbf{y} \in V,$$

$$b_{j_1, \dots, j_p}(\mathbf{y}) = \sum_{i_1 < \dots < i_p} a_{i_1, \dots, i_p}(f(\mathbf{y})) \cdot \frac{\partial(x_{i_1}, \dots, x_{i_p})}{\partial(y_{j_1}, \dots, y_{j_p})},$$

where $\frac{\partial(x_{i_1}, \dots, x_{i_p})}{\partial(y_{j_1}, \dots, y_{j_p})}$ denotes the Jacobian determinant.

Definition 1.9. A natural transformation $d : \wedge^p \rightarrow \wedge^{p+1}$ is given as follows: for any domain A , $d : \wedge^p(A) \rightarrow \wedge^{p+1}(A)$ is defined, for any $\eta = a(\mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p} \in \wedge^p(A)$, as

$$d\eta = \sum_i \frac{\partial a_{i_1, \dots, i_p}}{\partial x_i}(\mathbf{x}) dx_i \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}.$$

Then the naturality is obtained using a strait-forward computation.

A differential form is given in this context as follows.

Definition 1.10. Let (X, \mathcal{E}^X) be a differentiable or diffeological space, $\mathcal{E} = \mathcal{C}$ or \mathcal{D} .

(general): A differential p -form on X is a natural transformation $\omega : \mathcal{E}^X \rightarrow \wedge^p$ given by $\{\omega_A : \mathcal{E}^X(A) \rightarrow \wedge^p(A) ; A \in \mathcal{O}(\text{Domain})\}$ of contravariant functors $\mathcal{E}^X, \wedge^p : \text{Domain} \rightarrow \text{Set}$, in other words, ω satisfies $f^*(\omega_B(P)) = f^* \circ \omega_B(P) = \omega_A \circ f^*(P) = \omega_A(P \circ f)$ for any map $f : A \rightarrow B$ in **Domain** and a plot $P \in \mathcal{E}^X(B)$. The set of differential p -forms on X is denoted by $\Omega_{\mathcal{E}}^p(X)$ or simply by $\Omega^p(X)$. We also denote $\Omega_{\mathcal{E}}^*(X) = \bigoplus_p \Omega_{\mathcal{E}}^p(X)$ or by $\Omega^*(X) = \bigoplus_p \Omega^p(X)$.

(with compact support): A differential p -form with compact support on X is a natural transformation $\omega =: \mathcal{E}^X \rightarrow \wedge^p(-)$ with a compact subset $K_\omega \subset X$ such that, for any $A \in \mathcal{O}(\text{Domain})$ and $P \in \mathcal{E}^X$, we have $\text{Supp } \omega_A(P) \subset P^{-1}(K_\omega)$. The

set of differential p -forms with compact support on X is denoted by $\Omega_{\mathcal{E}_c}^p(X)$ or simply by $\Omega_c^p(X)$. We also denote $\Omega_{\mathcal{E}_c}^*(X) = \bigoplus_p \Omega_{\mathcal{E}_c}^p(X)$ or $\Omega_c^*(X) = \bigoplus_p \Omega_c^p(X)$.

Example 1.11. We have $\Omega^*(\{*\}) \cong \mathbb{R}$ and $\Omega_c^*(\{*\}) \cong \mathbb{R}$.

Definition 1.12 (External derivative). The external derivative of a differential p -form ω on a differentiable/diffeological space X is a differential $p+1$ -form $d\omega$ given by $(d\omega)_A = d \circ \omega_A$ for any $A \in \mathcal{O}(\text{Domain})$. If, further we assume $\omega \in \Omega_c^p(X)$, we clearly have $d\omega \in \Omega_c^{p+1}(X)$. Thus the external derivative induces endomorphisms of $\Omega^*(X)$ and $\Omega_c^*(X)$.

The category of differentiable or diffeological spaces and differentiable maps is denoted by **Differentiable** or **Diffeology**, respectively. By [10], [5] and [1], we know **Differentiable** and **Diffeology** are cartesian closed, complete and cocomplete.

Definition 1.13. Let $f : (X, \mathcal{E}^X) \rightarrow (Y, \mathcal{E}^Y)$ be a differentiable map, $\mathcal{E} = \mathcal{C}$ or \mathcal{D} .

- (1) We obtain a homomorphism $f^\sharp : \Omega^p(Y) \rightarrow \Omega^p(X)$: let $\omega \in \Omega^p(Y)$. Then $(f^\sharp \omega)_A(P) = \omega_A(f \circ P)$ for any $P \in \mathcal{E}^X(A)$ and $A \in \mathcal{O}(\text{Domain})$.
- (2) If a differentiable map f is proper, then we have $f^\sharp(\Omega_c^p(Y)) \subset \Omega_c^p(X)$ by taking $K_{f^\sharp \omega} = f^{-1}(K_\omega)$ for any $\omega \in \Omega_c^p(Y)$.

Definition 1.14. For an inclusion $j : U \hookrightarrow X$ of an open set U into a weakly-separated differentiable/diffeological space X , a homomorphism $j_\sharp : \Omega_c^p(U) \rightarrow \Omega_c^p(X)$ is defined as follows: for any $\omega \in \Omega_c^p(U)$, $j_\sharp \omega \in \Omega_c^p(X)$ is given, for n -domain B and $Q \in \mathcal{E}^X(B)$, by

$$\begin{cases} (j_\sharp \omega)_B(Q)|_A = \omega_A(Q|_A), & \text{if } A \text{ is an open } n\text{-domain in } Q^{-1}(U), \\ (j_\sharp \omega)_B(Q)|_A = 0, & \text{if } A \text{ is an open } n\text{-domain in } B \setminus Q^{-1}(K_\omega) \end{cases}$$

with $K_{j_\sharp \omega} = K_\omega \subset U \subset X$. Here, $\{Q^{-1}(U), B \setminus Q^{-1}(K_\omega)\}$ is an open covering of B .

Remark 1.15. In Definition 1.14, the map j_\sharp induced from an inclusion $j : U \hookrightarrow X$ satisfies that $(j_\sharp \omega)_B(j \circ Q) = \omega_B(Q)$ for any $B \in \mathcal{O}(\text{Domain})$ and $Q \in \mathcal{E}^U(B)$.

Proposition 1.16. There is an isomorphism $\Phi : \Omega^0(X) \cong C^\infty(X, \mathbb{R})$ such that $\Phi(\omega) \circ f = \Phi(f^\sharp(\omega))$ for any $\omega \in \Omega^0(X)$ and $f \in C^\infty(Y, X)$.

Proof: Firstly, we define a homomorphism $\Phi : \Omega^0(X) \rightarrow \mathcal{M}(\text{Set})(X, \mathbb{R})$ by $\Phi(\omega)(x) = \omega_{\{*\}}(c_x)(*) \in \mathbb{R}$ for any $\omega \in \Omega^0(X)$ and $x \in X$. By definition, Φ clearly is a homomorphism.

Secondly, we show $\text{Im } \Phi \subset C^\infty(X, \mathbb{R})$. For any n -domain A and $P \in \mathcal{E}^X(A)$, we have $\omega_A(P) : A \rightarrow \wedge^0(T_n^*) = \mathbb{R}$. Hence for any $\mathbf{x} \in A$, we have $P \circ c_{\mathbf{x}} = c_x \in \mathcal{E}^X(\{*\})$ where $x = P(\mathbf{x}) \in X$, and hence we have $\omega_A(P)(\mathbf{x}) = \omega_A(P) \circ c_{\mathbf{x}}(*) = \omega_{\{*\}}(P \circ c_{\mathbf{x}})(*) = \omega_{\{*\}}(c_x)(*) = \Phi(\omega)(x) = \Phi(\omega) \circ P(\mathbf{x})$, $\mathbf{x} \in A$. Thus we have $\omega_A(P) = \Phi(\omega) \circ P$ for any

$A \in \mathcal{O}(\text{Domain})$ and $P \in \mathcal{E}^X(A)$, and hence $\Phi(\omega) : X \rightarrow \mathbb{R}$ is a differentiable map. Moreover, for any differentiable map $f : Y \rightarrow X$, we have $\Phi(f^\sharp \omega)(x) = (f^\sharp \omega)_{\{*\}}(c_x)(*) = \omega_{\{*\}}(f \circ c_x)(*) = \omega_{\{*\}}(c_{f(x)})(*) = \Phi(\omega) \circ f(x)$, and hence we obtain $\Phi(f^\sharp \omega) = \Phi(\omega) \circ f$.

Thirdly, by the formula $\omega_A(P) = \Phi(\omega) \circ P$ for any $A \in \mathcal{O}(\text{Domain})$ and $P \in \mathcal{E}^X(A)$, ω is completely determined by $\Phi(\omega)$, and hence Φ is a monomorphism.

Finally, for any differentiable map $f : X \rightarrow \mathbb{R}$, we have a 0-form ω by $\omega_A(P) = f \circ P$ for any $A \in \mathcal{O}(\text{Domain})$ and $P \in \mathcal{E}^X(A)$, which also implies $\Phi(\omega) = f$. Thus Φ is an epimorphism, and it completes the proof of the proposition. \square

Definition 1.17. Let $X = (X, \mathcal{E})$ be a differentiable/diffeological space, $\mathcal{E} = \mathcal{C}$ or \mathcal{D} .

de Rham cohomology: $H_{\mathcal{E}}^p(X) = \frac{Z_{\mathcal{E}}^p(X)}{B_{\mathcal{E}}^p(X)}$,

where $Z_{\mathcal{E}}^p(X) = \text{Ker } d \cap \Omega_{\mathcal{E}}^p(X)$ and $B_{\mathcal{E}}^p(X) = d(\Omega_{\mathcal{E}}^p(X))$.

de Rham cohomology with compact support: $H_{\mathcal{E}_c}^p(X) = \frac{Z_{\mathcal{E}_c}^p(X)}{B_{\mathcal{E}_c}^p(X)}$,

where $Z_{\mathcal{E}_c}^p(X) = \text{Ker } d \cap \Omega_{\mathcal{E}_c}^p(X)$ and $B_{\mathcal{E}_c}^p(X) = d(\Omega_{\mathcal{E}_c}^p(X))$.

From now on, we often abbreviate as $H^p(X) = H_{\mathcal{E}}^p(X)$, $H_c^p(X) = H_{\mathcal{E}_c}^p(X)$ and so on.

Remark 1.18. We have $H_{\mathcal{E}_c}^p(M) \cong H_{dR}^p(M)$ and $H_{\mathcal{E}_c}^p(M) \cong H_{dR_c}^p(M)$ for a manifold M , where we denote by $H_{dR}^p(M)$ ($H_{dR_c}^p(M)$) the de Rham cohomology (with compact support).

Proposition 1.19. Let (X, \mathcal{E}^X) and (Y, \mathcal{E}^Y) be differentiable/diffeological spaces.

- (1) For a differentiable map $f : X \rightarrow Y$, the homomorphism $f^\sharp : \Omega^*(Y) \rightarrow \Omega^*(X)$ induces a homomorphism $f^* : H^*(Y) \rightarrow H^*(X)$.
- (2) If a differentiable map $f : X \rightarrow Y$ is proper, then the homomorphism $f^\sharp : \Omega_c^*(Y) \rightarrow \Omega_c^*(X)$ induces a homomorphism $f^* : H_c^*(Y) \rightarrow H_c^*(X)$.

Theorem 1.20. The de Rham cohomologies determines contravariant functors $H_{\mathcal{C}}^* : \text{Differentiable} \rightarrow \text{GradedAlgebra}$ and $H_{\mathcal{D}}^* : \text{Diffeology} \rightarrow \text{GradedAlgebra}$.

Proposition 1.21. Let (X, \mathcal{E}^X) be a weakly-separated differentiable/diffeological space and U an open set in X . Then the homomorphism $j_\sharp : \Omega_c^*(U) \rightarrow \Omega_c^*(X)$ induced from the canonical inclusion $j : U \hookrightarrow X$ induces a homomorphism $j_* : H_c^*(U) \rightarrow H_c^*(X)$.

Theorem 1.22 ([5], [10]). If two differentiable maps $f_0, f_1 : X \rightarrow Y$ between differentiable/diffeological spaces are homotopic in $C_{\mathcal{E}}^\infty(X, Y)$, $\mathcal{E} = \mathcal{C}$ or \mathcal{D} , i.e., there is a differentiable map $f : I \rightarrow C_{\mathcal{E}}^\infty(X, Y)$ such that $f(t) = f_t$, $t = 0, 1$, then we obtain

$$f_0^* = f_1^* : H_{\mathcal{E}}^*(Y) \rightarrow H_{\mathcal{E}}^*(X).$$

Theorem 1.23. *By definition, we clearly have $H_{\mathcal{E}}^*(\coprod_{\alpha} X_{\alpha}) = \prod_{\alpha} H_{\mathcal{E}}^*(X_{\alpha})$, $\mathcal{E} = \mathcal{C}$ or \mathcal{D} .*

Example 1.24. *For a differentiable/diffeological space $(\{*\}, \mathcal{E}^*)$ with $\mathcal{E}^*(A) = \{c_*\}$ for any $A \in \mathcal{O}(\text{Domain})$, we have $H^0(X) = \Omega^0(X) = \mathbb{R}$ and $H^p(X) = \Omega^p(X) = 0$ if $p \neq 0$.*

2. MAYER-VIETORIS SEQUENCE FOR DIFFERENTIABLE SPACES

Definition 2.1 (partition of unity). *Let (X, \mathcal{E}^X) be a differentiable/diffeological space and \mathcal{U} an open covering of X . A set of 0-forms $\boldsymbol{\rho} = \{\rho^U; U \in \mathcal{U}\}$ is called a partition of unity belonging to \mathcal{U} , if, for any $A \in \mathcal{O}(\text{Domain})$ and $P \in \mathcal{E}^X(A)$, $\text{Supp } \rho_A^U(P) \subset P^{-1}(U)$ and $\sum_{U \in \mathcal{U}} \rho_A^U(\mathbf{x}) = 1$, $\mathbf{x} \in A$. If further there is a family $\{G_U; U \in \mathcal{U}\}$ of closed sets in X such that, $\text{Supp } \rho_A^U(P) \subset P^{-1}(G_U)$ for any A and P above, then we say that $\boldsymbol{\rho}$ is ‘normal’.*

The above definition of a partition of unity using the notion of 0-form first appeared in Izumida [8] which was essentially the same as the one in Haraguchi [6] using the notion of a differentiable function, since a differential 0-form is a differentiable function, if we adopt the usual definition of 0-form. We introduce a special kind of open coverings as follows.

Definition 2.2 (Nice covering). *Let X be a differentiable space. An open covering \mathcal{U} of X is nice, if there is a partition of unity $\{\rho_A^U : A \rightarrow I = [0, 1]; U \in \mathcal{U}\}$ belonging to \mathcal{U} , i.e., $\{\rho^U\}$ are differential 0-forms with $\text{Supp } \rho_A^U(P) = \text{Cl}(\rho_A^U(P)^{-1}(I \setminus \{0\})) \subset P^{-1}(U)$, $U \in \mathcal{U}$ satisfying $\sum_{U \in \mathcal{U}} \rho_A^U(P)(\mathbf{x}) = 1$ for any $\mathbf{x} \in A$, where $\rho_A^U(P)(\mathbf{x}) \neq 0$ for finitely many U .*

Theorem 2.3 (see [6] or [8]). *Let $\mathcal{U} = \{U_1, U_2\}$ be a nice open covering of a differentiable/diffeological space (X, \mathcal{E}^X) with a partition of unity $\{\rho^{(1)}, \rho^{(2)}\}$ belonging to \mathcal{U} . Then $i_t : U_1 \cap U_2 \hookrightarrow U_t$ and $j_t : U_t \hookrightarrow X$, $t = 1, 2$, induce homomorphisms $\psi^{\sharp} : \Omega^p(X) \rightarrow \Omega^p(U_1) \oplus \Omega^p(U_2)$ and $\phi^{\sharp} : \Omega^p(U_1) \oplus \Omega^p(U_2) \rightarrow \Omega^p(U_1 \cap U_2)$ by $\psi^{\sharp}(\omega) = i_1^{\sharp}\omega \oplus i_2^{\sharp}\omega$ and $\phi^{\sharp}(\eta_1 \oplus \eta_2) = j_1^{\sharp}\eta_1 - j_2^{\sharp}\eta_2$, and the following sequence is exact.*

$$\begin{aligned} H^0(X) \rightarrow \cdots \rightarrow H^p(X) &\xrightarrow{\psi^*} H^p(U_1) \oplus H^p(U_2) \xrightarrow{\phi^*} H^p(U_1 \cap U_2) \\ &\rightarrow H^{p+1}(X) \xrightarrow{\psi^*} H^{p+1}(U_1) \oplus H^{p+1}(U_2) \xrightarrow{\phi^*} H^{p+1}(U_1 \cap U_2) \rightarrow \cdots, \end{aligned}$$

where ψ^* and ϕ^* are induced from ψ^{\sharp} and ϕ^{\sharp} .

Proof: Let $U_0 = U_1 \cap U_2$. We show that the following sequence is short exact.

$$0 \longrightarrow \Omega^p(X) \xrightarrow{\psi^{\sharp}} \Omega^p(U_1) \oplus \Omega^p(U_2) \xrightarrow{\phi^{\sharp}} \Omega^p(U_0) \longrightarrow 0.$$

(**exactness at $\Omega^p(X)$**): Assume $\psi^{\sharp}(\omega) = 0$, and so $j_t^{\sharp}\omega = 0$ for $t=1, 2$. For any $A \in \mathcal{O}(\text{Domain})$ and $P \in \mathcal{E}^X(A)$, we define $P_t : P^{-1}(U_t) \rightarrow U_t$, $t=1, 2$ by $P_t(\mathbf{x}) = P(\mathbf{x})$

for any $\mathbf{x} \in P^{-1}(U_t)$, so that $P|_{P^{-1}(U_t)} = j_t \circ P_t$ for $t=1, 2$. Then, for any $\mathbf{x} \in A$, there is an open subset $A_{\mathbf{x}} \in \mathcal{O}(\text{Domain})$ of A such that $\mathbf{x} \in A_{\mathbf{x}} \subset P^{-1}(U_t)$ for $t=1$ or 2 . In each case, we have $\omega_A(P)|_{A_{\mathbf{x}}} = \omega_{A_{\mathbf{x}}}(P|_{A_{\mathbf{x}}}) = \omega_{A_{\mathbf{x}}}(P|_{P^{-1}(U_t)}|_{A_{\mathbf{x}}}) = \omega_{A_{\mathbf{x}}}(j_t \circ P_t|_{A_{\mathbf{x}}}) = (j_t^\# \omega)_{A_{\mathbf{x}}}(P_t|_{A_{\mathbf{x}}}) = 0$, and hence $\omega_A(P)|_{A_{\mathbf{x}}} = 0$ for any $\mathbf{x} \in A$. Thus $\omega_A(P) = 0$ for any A and P , which implies that $\omega = 0$. Thus ψ^\sharp is monic.

(exactness at $\Omega^p(U_1) \oplus \Omega^p(U_2)$): Assume $\phi^\sharp(\eta^{(1)} \oplus \eta^{(2)}) = 0$, and so $i_1^\# \eta^{(1)} = i_2^\# \eta^{(2)}$.

Then we construct $\omega \in \Omega^p(X)$ as follows. For any $A \in \mathcal{O}(\text{Domain})$ and $P \in \mathcal{E}^X(A)$, $\{P^{-1}(U_t); t=1, 2\}$ is an open covering of A , and for $t=0, 1, 2$ we obtain $P_t : P^{-1}(U_t) \rightarrow U_t$ given by $P_t(\mathbf{x}) = P(\mathbf{x})$ for any $\mathbf{x} \in P^{-1}(U_t)$, so that $P_t|_{P^{-1}(U_0)} = i_t \circ P_0$ for $t=1, 2$. For any $\mathbf{x} \in A$, there is an open subset $A_{\mathbf{x}} \in \mathcal{O}(\text{Domain})$ of A such that $\mathbf{x} \in A_{\mathbf{x}} \subset P^{-1}(U_t)$ for $t=1$ or 2 . Using it, we define $\omega_A(P)(\mathbf{x}) = \eta_{A_{\mathbf{x}}}^{(t)}(P|_{A_{\mathbf{x}}})(\mathbf{x})$ for any $\mathbf{x} \in A$. In case when $A_{\mathbf{x}} \subset A_0 = A_1 \cap A_2$, we have $\eta_{A_{\mathbf{x}}}^{(1)}(P_1|_{A_{\mathbf{x}}}) = \eta_{A_{\mathbf{x}}}^{(1)}(i_1 \circ P_0|_{A_{\mathbf{x}}}) = (i_1^\# \eta^{(1)})_{A_{\mathbf{x}}}(P_0|_{A_{\mathbf{x}}}) = (i_2^\# \eta^{(2)})_{A_{\mathbf{x}}}(P_0|_{A_{\mathbf{x}}}) = \eta_{A_{\mathbf{x}}}^{(2)}(i_2 \circ P_0|_{A_{\mathbf{x}}}) = \eta_{A_{\mathbf{x}}}^{(2)}(P_2|_{A_{\mathbf{x}}})$, and hence $\eta_{A_{\mathbf{x}}}^{(1)}(P_1|_{A_{\mathbf{x}}}) = \eta_{A_{\mathbf{x}}}^{(2)}(P_2|_{A_{\mathbf{x}}})$. It implies that ω is well-defined and $\psi^\sharp(\omega) = \eta^{(1)} \oplus \eta^{(2)}$. The converse is clear and we obtain $\text{Ker } \phi^\sharp = \text{Im } \psi^\sharp$.

(exactness at $\Omega^p(U_0)$): Assume $\kappa \in \Omega^p(U_0)$. Then we define $\kappa^{(t)} \in \Omega^p(U_t)$, $t=1, 2$ defined as follows. For any $A_t \in \mathcal{O}(\text{Domain})$ and a plot $P_t : A_t \rightarrow U_t$, we define $\kappa_{A_t}^{(t)}(P_t)(\mathbf{x})$ by $(-1)^{t-1} \rho_{A_t}^{3-t}(P_t)(\mathbf{x}) \cdot \kappa_{A_t}(P_t)(\mathbf{x})$ if $\mathbf{x} \in P_t^{-1}(U_{3-t})$ and by 0 if $\mathbf{x} \notin P_t^{-1}(\text{Supp } \rho_{A_t}^{3-t}(P_t))$. Then we see that $\kappa^{(t)}$ is well-defined differential p -form on U_t and $i_1^\# \kappa^{(1)} - i_2^\# \kappa^{(2)} = \kappa$, and hence $\phi^\sharp(\kappa^{(1)} \oplus \kappa^{(2)}) = \kappa$. Thus ϕ^\sharp is an epimorphism.

Since ψ^\sharp and ϕ^\sharp are clearly cochain maps, we obtain the desired long exact sequence. \square

Let us turn our attention to the differential forms with compact support.

Theorem 2.4 (see [6] or [8]). *Let (X, \mathcal{E}^X) be a weakly-separated differentiable/diffeological space and $\mathcal{U} = \{U_1, U_2\}$ a nice open covering of X with a normal partition of unity $\{\rho^{(1)}, \rho^{(2)}\}$ belonging to \mathcal{U} . Then $i_t : U_1 \cap U_2 \hookrightarrow U_t$ and $j_t : U_t \hookrightarrow X$, $t=1, 2$, induce homomorphisms $\phi_\sharp : \Omega_c^p(U_1 \cap U_2) \rightarrow \Omega_c^p(U_1) \oplus \Omega_c^p(U_2)$ and $\psi_\sharp : \Omega_c^p(U_1) \oplus \Omega_c^p(U_2) \rightarrow \Omega_c^p(X)$ by $\phi_\sharp(\omega) = i_{1\#} \omega \oplus i_{2\#} \omega$ and $\psi_\sharp(\eta_1 \oplus \eta_2) = j_{1\#} \eta_1 - j_{2\#} \eta_2$, and the following sequence is exact.*

$$\begin{aligned} H_c^0(U_1 \cap U_2) &\rightarrow \cdots \rightarrow H_c^p(U_1 \cap U_2), \xrightarrow{\phi_*} H_c^p(U_1) \oplus H_c^p(U_2) \xrightarrow{\psi_*} H_c^p(X) \\ &\rightarrow H_c^{p+1}(U_1 \cap U_2) \xrightarrow{\phi_*} H_c^{p+1}(U_1) \oplus H_c^{p+1}(U_2) \xrightarrow{\psi_*} H_c^{p+1}(X) \rightarrow \cdots, \end{aligned}$$

where ψ_* and ϕ_* are induced from ψ_\sharp and ϕ_\sharp .

Proof: Let $U_0 = U_1 \cap U_2$. We show that the following sequence is short exact.

$$0 \longrightarrow \Omega_c^p(U_0) \xrightarrow{\phi_\sharp} \Omega_c^p(U_1) \oplus \Omega_c^p(U_2) \xrightarrow{\psi_\sharp} \Omega_c^p(X) \longrightarrow 0.$$

(exactness at $\Omega_c^p(U_0)$): Assume $\phi_{\natural}(\omega) = 0$. Then $i_{1\sharp}(\omega) = i_{2\sharp}(\omega) = 0$. Since $i_{1\sharp}(\omega)$ is an extension of ω , we obtain $\omega = 0$. Thus ϕ_{\natural} is a monomorphism.

(exactness at $\Omega_c^p(U_1) \oplus \Omega_c^p(U_2)$): Assume $\psi_{\natural}(\eta^{(1)} \oplus \eta^{(2)}) = 0$. By definition, we have $j_{1\sharp}(\eta^{(1)}) = j_{2\sharp}(\eta^{(2)})$. For any $A \in \mathcal{O}(\text{Domain})$ and $P \in \mathcal{E}^X(A)$, we have $j_{1\sharp}(\eta^{(1)})_A(P) = j_{2\sharp}(\eta^{(2)})_A(P)$. So, for any $B \in \mathcal{O}(\text{Domain})$ and a plot $Q : B \rightarrow U_0$, $\eta_B^{(1)}(i_1 \circ Q) = j_{1\sharp}^{\#} \eta_B^{(1)}(j_1 \circ i_1 \circ Q) = j_{2\sharp}^{\#} \eta_B^{(2)}(j_2 \circ i_2 \circ Q) = \eta_B^{(2)}(i_2 \circ Q)$. So we define $\eta^{(0)} \in \Omega^p(U_0)$ by $\eta_B^{(0)}(Q) = \eta_B^{(1)}(i_1 \circ Q) = \eta_B^{(2)}(i_2 \circ Q)$. On the other hand, $K_{j_{t\sharp}\eta^{(t)}} = K_{\eta^{(t)}}$ by definition, and hence we obtain $\text{Supp } \eta_B^{(0)}(Q) = \text{Supp } \eta_B^{(1)}(i_1 \circ Q) = \text{Supp } \eta_B^{(2)}(i_2 \circ Q) \subset Q^{-1}(K_{\eta^{(1)}} \cap K_{\eta^{(2)}})$. Then $\eta^{(0)} \in \Omega_c^p(U_0)$ for $K_{\eta^{(0)}} = K_{\eta^{(1)}} \cap K_{\eta^{(2)}}$ is compact.

(exactness at $\Omega_c^p(X)$): Assume $\kappa \in \Omega_c^p(X)$. For any $A_t \in \mathcal{O}(\text{Domain})$ and a plot $P_t : A_t \rightarrow U_t$, we define $\kappa_{A_t}^{(t)}(P_t)(\mathbf{x})$ by $(-1)^{t-1} \rho_{A_t}^{(t)}(P_t)(\mathbf{x}) \cdot \kappa_{A_t}(j_t \circ P_t)(\mathbf{x})$ if $\mathbf{x} \in P_t^{-1}(U_0)$ and by 0 if $\mathbf{x} \notin \text{Supp } \rho_{A_t}^{(t)}(P_t)$. Then $\kappa^{(t)}$ is a well-defined differential p -form on U_t with compact support $K_{\kappa^{(t)}} = K_{\kappa} \cap G_{U_t}$ in U_t and $j_1^{\#} \kappa^{(1)} - j_2^{\#} \kappa^{(2)} = \kappa$, and hence we have $\psi_{\natural}(\kappa^{(1)} \oplus \kappa^{(2)}) = \kappa$. Thus ψ_{\natural} is an epimorphism.

Since ϕ_{\natural} and ψ_{\natural} are clearly cochain maps, we obtain the desired long exact sequence. \square

3. CUBE CATEGORY

Definition 3.1. a concrete monoidal site \square is defined as follows:

Object: $\mathcal{O}(\square) = \{\underline{0}, \underline{1}, \underline{2}, \dots\} \approx \mathbb{N}_0$, $\underline{n} = \square_L^n := \square^n \cap L$,

where $\square^n = \{(t_1, \dots, t_n) ; 0 \leq t_1, \dots, t_n \leq 1\}$ and $L = \mathbb{Z}^n \subset \mathbb{R}^n$ is an integral lattice.

Morphism: $\mathcal{M}(\square)$ is generated by the following sets of morphisms.

boundary: $\partial_i^{\epsilon} : \underline{n} \rightarrow \underline{n+1}$, $\epsilon \in \dot{I} = \{0, 1\}$, $1 \leq i \leq n+1$, $n \geq 0$, given by

$$\partial_i^{\epsilon}(\mathbf{t}) = (t_1, \dots, t_{i-1}, \epsilon, t_{i+1}, \dots, t_n) \text{ for } \mathbf{t} = (t_1, \dots, t_n) \in \square_L^n,$$

degeneracy: $\varepsilon_i : \underline{n+1} \rightarrow \underline{n}$, $1 \leq i \leq n+1$, $n \in \mathbb{N}_0$ given by

$$\varepsilon_i(\mathbf{t}) = (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{n+1}), \quad \mathbf{t} = (t_1, \dots, t_{n+1}) \in \square_L^{n+1},$$

which satisfies the following relations.

$$\begin{aligned} (1) \quad \partial_j^{\epsilon'} \circ \partial_i^{\epsilon} &= \begin{cases} \partial_i^{\epsilon} \circ \partial_{j-1}^{\epsilon'} & \text{if } i < j \\ \partial_{i+1}^{\epsilon} \circ \partial_j^{\epsilon'} & \text{if } i \geq j \end{cases} & (2) \quad \varepsilon_j \circ \varepsilon_i &= \begin{cases} \varepsilon_i \circ \varepsilon_{j+1} & \text{if } i \leq j \\ \varepsilon_{i-1} \circ \varepsilon_j & \text{if } i > j \end{cases} \\ (3) \quad \partial_j^{\epsilon'} \circ \varepsilon_i &= \begin{cases} \varepsilon_{i+1} \circ \partial_j^{\epsilon'} & \text{if } i \geq j \\ \varepsilon_i \circ \partial_{j+1}^{\epsilon'} & \text{if } i < j \end{cases} & (4) \quad \varepsilon_j \circ \partial_i^{\epsilon} &= \begin{cases} \partial_{i-1}^{\epsilon} \circ \varepsilon_j & \text{if } i > j \\ \partial_i^{\epsilon} \circ \varepsilon_{j-1} & \text{if } i < j \\ \text{id} & \text{if } i = j \end{cases} \end{aligned}$$

Since $\square_L^n = \square^n \cap L \subset \mathbb{R}^n$, we can extend the boundaries and the degeneracies as smooth maps $\partial_i^\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ and $\varepsilon_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$. Let $\square : \square \rightarrow \mathbf{Convex}$ be the covariant functor defined by $\square(\underline{n}) = \square^n$, $\square(\partial_i^\varepsilon) = \partial_i^\varepsilon|_{\square^n} : \square^n \rightarrow \square^{n+1}$ and $\square(\varepsilon_i) = \varepsilon_i|_{\square^{n+1}} : \square^{n+1} \rightarrow \square^n$.

Remark 3.2. *There is a smooth relative homeomorphism $\pi_n : (\square^n, \partial\square^n) \rightarrow (\triangle^n, \partial\triangle^n)$ given by $\pi_n(t_1, \dots, t_n) = (s_1, \dots, s_n)$, $s_k = t_k \cdots t_n$, where the standard n -simplex \triangle^n is assumed to be as $\triangle^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n; 0 = x_0 \leq x_1 \leq \dots \leq x_n \leq x_{n+1} = 1\}$.*

According to [1], there is a natural embedding $ch : \mathbf{Diffeology} \rightarrow \mathbf{Differentiable}$. So, from now on, we deal mainly with differentiable spaces, rather than diffeological spaces. We denote $\mathcal{E}_{\square}^X = \mathcal{E}^X \circ \square$ and $\wedge_{\square}^p = \wedge^p \circ \square$, and a plot in $\mathcal{E}_{\square}^X(\underline{n}) = \mathcal{E}^X(\square^n)$ is called an n -plot.

Let $X = (X, \mathcal{E}^X)$ be a differentiable space. Then we denote $\Sigma_n(X) = \mathcal{E}^X(\square^n)$ the set of n -plots. Let $\Gamma_n(X)$ be the free abelian group generated by $\Sigma_n(X)$ and $\Gamma^n(X, R) = \text{Hom}(\Gamma_n(X); R)$, where R is a commutative ring with unit. Then $\Gamma^*(X; R)$ is a cochain complex and we obtain a smooth version of cubical singular cohomology $H^*(X, R)$ in a canonical manner, which satisfies axioms of cohomology theories such as additivity, dimension and homotopy axioms together with a Mayer-Vietoris exact sequence.

4. CUBICAL DE RHAM COHOMOLOGY

We introduce a version of a differential form by using \mathcal{E}_{\square}^X and \wedge_{\square}^p .

Definition 4.1 (cubical differential form). *A cubical differential form on a differentiable space X is a natural transformation $\omega : \mathcal{E}_{\square}^X \rightarrow \wedge_{\square}^p$ of contravariant functors $\square \rightarrow \mathbf{Set}$. We denote $\omega = \{\omega_{\underline{n}}; n \geq 0\}$, where $\omega_{\underline{n}} : \mathcal{E}^X(\square^n) \rightarrow \wedge^p(\square^n)$. The set of cubical differential forms on X is denoted by $\Omega_{\square}^p(X)$ and $\Omega_{\square}^*(X) = \bigoplus_p \Omega_{\square}^p(X)$.*

We denote by $\square^* : \Omega_{\mathcal{C}}^p(X) \rightarrow \Omega_{\square}^p(X)$ the natural map induced from $\square : \square \rightarrow \mathbf{Convex}$.

Theorem 4.2. *The map $\square^* : \Omega_{\mathcal{C}}^p(X) \rightarrow \Omega_{\square}^p(X)$ is monic.*

Proof: Assume that $\omega \in \Omega_{\mathcal{C}}^p(X)$ satisfies $\square^*(\omega) = 0 : \mathcal{E}_{\square}^X \rightarrow \wedge_{\square}^p$.

By induction on n , we show $\omega_A = 0$ for any convex n -domain A .

($n = 0$) In this case, we have $\Omega_{\mathcal{C}}^0(X) = \Omega_{\square}^0(X)$ and $\omega_{\text{points}} = 0$.

($n > 0$) Let $P : A \rightarrow X$ be a plot of X , where A is a convex n -domain. For any element $u \in \text{Int } A$, there is a small simplex $\square_u^n \subset \text{Int } A$ such that $\text{Int } \square_u^n \ni u$. Then there is a linear diffeomorphism $\phi : \square^n \approx \square_u^n$. Hence $P \circ \phi \in C_c^\infty(\square^n, X)$ and we obtain

$$0 = \square^*(\omega)_n(P \circ \phi) = \omega_{\square^n}(P \circ \phi) = \phi^*(\omega_{\square_u^n}(P|_{\square_u^n})) = \phi^*(\omega_A(P)|_{\square_u^n}).$$

Since ϕ is a diffeomorphism, we have $\omega_A(P)|_{\square_u^n} = 0$ for any $u \in \text{Int } A$. Thus we obtain $\omega_A(P) = 0$ on $\text{Int } A$. Since $\omega_A(P)$ is continuous, $\omega_A(P) = 0$ on A . \square

A differentiable map induces a homomorphism of cubical differential forms as follows:

Definition 4.3. Let $f : X \rightarrow Y$ be a differentiable map between differentiable spaces $X = (X, \mathcal{E}^X)$ and $Y = (Y, \mathcal{E}^Y)$.

(1) We obtain a homomorphism $f^\# : \Omega_{\square}^p(Y) \rightarrow \Omega_{\square}^p(X)$: let $\omega \in \Omega_{\square}^p(Y)$. Then

$$(f^\# \omega_{\underline{n}})(P) = \omega_{\underline{n}}(f \circ P) \quad \text{for any } P \in \mathcal{E}_{\square}^X(\underline{n}), n \geq 0.$$

(2) If a differentiable map f is proper, then we have $f^\#(\Omega_{\square_c}^p(Y)) \subset \Omega_{\square_c}^p(X)$ by taking $K_{f^\# \omega} = f^{-1}(K_\omega)$ for any $\omega \in \Omega_{\square_c}^p(Y)$.

Definition 4.4 (External derivative). Let $X = (X, \mathcal{E})$ be a differentiable space. The external derivative $d : \Omega_{\square}^p(X) \rightarrow \Omega_{\square}^{p+1}(X)$ is defined as follows.

$$(d\omega)_{\underline{n}}(P) = d(\omega_{\underline{n}}(P)) \quad \text{for an } n\text{-plot } P \in \mathcal{E}_{\square}(\underline{n}) = \mathcal{E}(\square^n).$$

Definition 4.5. Let $X = (X, \mathcal{E})$ be a differentiable space.

Cubical de Rham cohomology: $H_{\square}^p(X) = \frac{Z_{\square}^p(X)}{B_{\square}^p(X)}$,

where $Z_{\square}^p(X) = \text{Ker } d \cap \Omega_{\square}^p(X)$ and $B_{\square}^p(X) = d(\Omega_{\square}^p(X))$.

Cubical de Rham cohomology with compact support: $H_{\square_c}^p(X) = \frac{Z_{\square_c}^p(X)}{B_{\square_c}^p(X)}$,

where $Z_{\square_c}^p(X) = \text{Ker } d \cap \Omega_{\square_c}^p(X)$ and $B_{\square_c}^p(X) = d(\Omega_{\square_c}^p(X))$.

Example 4.6. Let $X = (X, \mathcal{E}^X)$ be a differentiable space with $X = \{*\}$ one-point-set. Then we have $H_{\square}^p(\{*\}) = \mathbb{R}$ if $p = 0$ and 0 otherwise.

Proposition 4.7. Let $X = (X, \mathcal{E}^X)$ and $Y = (Y, \mathcal{E}^Y)$ be differentiable spaces.

(1) For a differentiable map $f : X \rightarrow Y$, the homomorphism $f^\# : \Omega_{\square}^*(Y) \rightarrow \Omega_{\square}^*(X)$ induces a homomorphism $H_{\square}^*(Y) \rightarrow H_{\square}^*(X)$.

(2) If a differentiable map $f : X \rightarrow Y$ is proper, then the homomorphism $f^\# : \Omega_{\square_c}^*(Y) \rightarrow \Omega_{\square_c}^*(X)$ induces a homomorphism $f^* : H_{\square_c}^*(Y) \rightarrow H_{\square_c}^*(X)$.

Theorem 4.8. By definition, we clearly have $H_{\square}^*(\coprod_{\alpha} X_{\alpha}) = \prod_{\alpha} H_{\square}^*(X_{\alpha})$.

Theorem 4.9. H_{\square}^* is a contravariant functor from Differentiable to GradedAlgebra.

5. HOMOTOPY INVARIANCE OF CUBICAL DE RHAM COHOMOLOGY

Let $f_0, f_1 : X \rightarrow Y$ be homotopic differentiable maps between differentiable spaces $X = (X, \mathcal{E}^X)$ and $Y = (Y, \mathcal{E}^Y)$. Then there is a plot $f : I \rightarrow C_c^\infty(X, Y)$ with $f(t) = f_t$ for $t = 0, 1$. In particular, for any n -plot $P : \square^n \rightarrow X$, $f \cdot P : \square^{n+1} = I \times \square^n \xrightarrow{f \cdot P} Y$ is an

$n+1$ -plot. Then, we obtain a homomorphism $D_f : \Omega_{\square}^p(Y) \rightarrow \Omega_{\square}^{p-1}(X)$ as follows: for any cubical differential p -form $\omega : \mathcal{E}_{\square}^Y \rightarrow \wedge_{\square}^p$ on Y , a $p-1$ -form $D_f(\omega) : \mathcal{E}_{\square}^X \rightarrow \wedge_{\square}^{p-1}$ on X is defined by the following formula.

$$D_f(\omega)_{\underline{n}}(P) = \int_I \omega_{\underline{n+1}}(f \cdot P) : \square^n \rightarrow \wedge^{p-1}(T_n^*),$$

$$\left[\int_I \omega_{\underline{n+1}}(f \cdot P) \right] (\mathbf{x}) = \sum_{i_2, \dots, i_p} \int_0^1 a_{i_2, \dots, i_p}(t, \mathbf{x}) dt \cdot dx_{i_2} \wedge \dots \wedge dx_{i_p},$$

where we assume $\omega_{\underline{n+1}}(f \cdot P) = \sum_{i_2, \dots, i_p} a_{i_2, \dots, i_p}(t, \mathbf{x}) dt \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p} + \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(t, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p}$, $(t, \mathbf{x}) \in I \times \square^n = \square^{n+1}$ and $T_{n+1}^* = \mathbb{R} dt \oplus \bigoplus_{i=1}^n \mathbb{R} dx_i$.

Lemma 5.1. *For any ω , we obtain $dD(\omega)_{\underline{n}} + D(d\omega)_{\underline{n}} = f_1^{\sharp}\omega_{\underline{n}} - f_0^{\sharp}\omega_{\underline{n}}$. Thus, if $d\omega = 0$, then $f_0^{\sharp}\omega$ is cohomologous to $f_1^{\sharp}\omega$.*

Proof: First, let $\omega_{\underline{n+1}}(f \cdot P) = \sum_{i_2, \dots, i_p} a_{i_2, \dots, i_p}(t, \mathbf{x}) dt \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p} + \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(t, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p}$. Let $\text{in}_t : \square^n \rightarrow I \times \square^n$ be the inclusion defined by $\text{in}_t(\mathbf{x}) = (t, \mathbf{x})$ for $t = 0, 1$. Since $(f \cdot P) \circ \text{in}_t = f_t \circ P$ for $t = 0, 1$, we have $(f_t^{\sharp}\omega_{\underline{n}})(P) = \omega_{\underline{n}}(f_t \circ P) = \omega_{\underline{n}}((f \cdot P) \circ \text{in}_t) = \text{in}_t^* \omega_{\underline{n+1}}(f \cdot P) = \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(t, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p}$ for $t = 0, 1$, $\mathbf{x} \in \square^n$.

Second, by definition, we have $d\omega_{\underline{n+1}}(f \cdot P) = \sum_i \sum_{i_2, \dots, i_p} \frac{\partial a_{i_2, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dx_i \wedge dt \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p} + \sum_{i_1, \dots, i_p} \frac{\partial b_{i_1, \dots, i_p}}{\partial t}(t, \mathbf{x}) dt \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p} + \sum_i \sum_{i_2, \dots, i_p} \frac{\partial b_{i_1, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dx_i \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}$, and hence we obtain $D(d\omega)_{\underline{n}}(P) = -\sum_i \sum_{i_2, \dots, i_p} \int_I \frac{\partial a_{i_2, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dt \cdot dx_i \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p} + \sum_{i_1, \dots, i_p} \int_I \frac{\partial b_{i_1, \dots, i_p}}{\partial t}(t, \mathbf{x}) dt \cdot dx_{i_1} \wedge \dots \wedge dx_{i_p}$, $(t, \mathbf{x}) \in I \times \square^n$.

Third, we have $D_f(\omega)_{\underline{n}}(P) = \sum_{i_2, \dots, i_p} \int_I a_{i_2, \dots, i_p}(t, \mathbf{x}) dt \cdot dx_{i_2} \wedge \dots \wedge dx_{i_p}$, and hence we obtain $dD_f(\omega)_{\underline{n}}(P) = \sum_i \sum_{i_2, \dots, i_p} \int_I \frac{\partial a_{i_2, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dt \cdot dx_i \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p}$, $(t, \mathbf{x}) \in I \times \square^n$.

Hence $[dD_f(\omega)_{\underline{n}}(P) + D_f(d\omega)_{\underline{n}}(P)](\mathbf{x}) = \sum_{i_1, \dots, i_p} \int_I \frac{\partial b_{i_1, \dots, i_p}}{\partial t}(t, \mathbf{x}) dt \cdot dx_{i_1} \wedge \dots \wedge dx_{i_p} = \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(1, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p} - \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(0, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p}$, $\mathbf{x} \in \square^n$. Thus we obtain $dD_f(\omega)_{\underline{n}}(P) + D_f(d\omega)_{\underline{n}}(P) = (f_1^{\sharp}\omega_{\underline{n}})(P) - (f_0^{\sharp}\omega_{\underline{n}})(P)$, which implies the lemma. \square

It immediately implies the following theorem.

Theorem 5.2. *If two differentiable maps $f_0, f_1 : X \rightarrow Y$ between differentiable spaces are homotopic in $C_c^\infty(X, Y)$, then they induce the same homomorphism*

$$f_0^* = f_1^* : H_{\square}^*(Y) \rightarrow H_{\square}^*(X).$$

6. HUREWICZ HOMOMORPHISM

First, we give a definition of paths and fundamental groupoid of a differentiable space.

Definition 6.1. *In this paper, a path from $a \in X$ to $b \in X$ in a differentiable space X means a differentiable map $\ell : I \rightarrow X$ such that $\ell(0) = a$ and $\ell(1) = b$. We denote by $\pi_0(X)$ the set of path-connected components of X , as usual.*

Definition 6.2. *Let \mathbf{Cat} be the category of all small categories. The fundamental groupoid functor $\pi_1 : \mathbf{Differentiable} \rightarrow \mathbf{Cat}$ is as follows:*

- (1) *For a differentiable space X , the small category $\pi_1(X)$ is defined by $\mathcal{O}(\pi_1(X)) = X$ and $\mathcal{M}(\pi_1(X))(x_0, x_1)$ is the set of homotopy classes of all differentiable maps $\ell : I \rightarrow X$ with $\ell(0) = x_0$ and $\ell(1) = x_1$ for any $x_0, x_1 \in X$.*
- (2) *For a differentiable map $f : Y \rightarrow X$, the functor $f_* : \pi_1(Y) \rightarrow \pi_1(X)$ is defined by $f_* = f : Y \rightarrow X$ and $f_*([\ell]) = [f \circ \ell]$ for any $[\ell] \in \pi_1(Y)$.*

Definition 6.3. *The functor $\mathbb{R} : \mathbf{Differentiable} \rightarrow \mathbf{Cat}$ is defined as follows:*

- (1) *For a differentiable space X , the small category $\mathbb{R}(X)$ is defined by $\mathcal{O}(\mathbb{R}(X)) = X$ and $\mathcal{M}(\mathbb{R}(X))(x_0, x_1) = \mathbb{R}$ for any $x_0, x_1 \in X$, and the composition is given by addition of real numbers.*
- (2) *For a differentiable map $f : Y \rightarrow X$, the functor $f_* : \mathbb{R}(Y) \rightarrow \mathbb{R}(X)$ is defined by $f_* = f : Y \rightarrow X$ and $f_* = \text{id} : \mathbb{R} \rightarrow \mathbb{R}$.*

Definition 6.4. *The Hurewicz homomorphism $\rho : Z_{\square}^1(X) \rightarrow \text{Hom}(\pi_1(X), \mathbb{R}(X))$ (the set of functors) is defined for any $\omega \in Z_{\square}^1(X)$ by $\rho(\omega)(x) = x$ for any $x \in \mathcal{O}(\pi_1(X)) = X$ and $\rho(\omega)([\ell]) = \int_I \omega_1(\ell)$ for any $[\ell] \in \mathcal{M}(\pi_1(X))$, which is natural, in other words, the diagram below is commutative for any differentiable map $f : Y \rightarrow X$ between differentiable spaces.*

$$\begin{array}{ccc}
 Z_{\square}^1(X) & \xrightarrow{\rho} & \text{Hom}(\pi_1(X), \mathbb{R}(X)) \\
 \downarrow f^* & & \downarrow \text{Hom}(f_*, \text{id}) \\
 Z_{\square}^1(Y) & \xrightarrow{\rho} & \text{Hom}(\pi_1(Y), \mathbb{R}(Y))
 \end{array}$$

(well-defined) Let $\ell_0 \sim \ell_1$ with $\ell_t(\epsilon) = x_{\epsilon} \in X$, $t = 0, 1$ and $\epsilon = 0, 1$. Then there is a 2-plot $\hat{\ell} : \square^2 \rightarrow X$ such that $\hat{\ell}(\epsilon, s) = \ell_{\epsilon}(s)$ and $\hat{\ell}(t, \epsilon) = x_{\epsilon}$ for $\epsilon = 0, 1$. Hence we have $\hat{\ell} \circ \partial_1^{\epsilon} = \ell_{\epsilon}$,

$\epsilon = 0, 1$ and $\hat{\ell} \circ \partial_2^\epsilon = c_{x_\epsilon} = c_{x_\epsilon} \circ \varepsilon_1$. Let $\omega_{\underline{2}}(\hat{\ell}) = a(t, s) dt + b(t, s) ds \in \wedge^1(\square^2)$. Then we have $\omega_{\underline{2}}(\ell_\epsilon) = \omega_{\underline{2}}(\hat{\ell} \circ \partial_1^\epsilon) = \partial_1^{\epsilon*} \omega_{\underline{2}}(\hat{\ell}) = b(\epsilon, s) ds$, $\epsilon = 0, 1$. Similarly, $0 = \varepsilon_1^* \omega_*(c_{x_\epsilon}) = \omega_{\underline{2}}(c_{x_\epsilon} \circ \varepsilon_1) = \omega_{\underline{2}}(\hat{\ell} \circ \partial_2^\epsilon) = \partial_2^{\epsilon*} \omega_{\underline{2}}(\hat{\ell}) = a(t, \epsilon) dt$ which implies $a(t, \epsilon) = 0$, $\epsilon = 0, 1$. On the other hand by Green's formula, we obtain that $\int_{\partial \square^2} (\omega_{\underline{2}}(\hat{\ell})|_{\partial \square^2}) = \int_{\square^2} d\omega = 0$, since ω is a closed form. Then it follows that $\int_{\{1\} \times I} (\omega_{\underline{2}}(\hat{\ell})|_{\{1\} \times I}) - \int_{\{0\} \times I} (\omega_{\underline{2}}(\hat{\ell})|_{\{0\} \times I}) = 0$, and hence $\int_I \omega_{\underline{1}}(\ell_1) = \int_I \omega_{\underline{1}}(\ell_0)$, and ρ is well-defined. The additivity of ρ is clear by definition.

(naturality) Let $f : Y \rightarrow X$ be a differentiable map. Then f induces both $f^* : Z_{\underline{\square}}^1(X) \rightarrow Z_{\underline{\square}}^1(Y)$ and $f_* : \pi_1(Y) \rightarrow \pi_1(X)$. The latter homomorphism induces

$$\text{Hom}(f_*, \text{id}) : \text{Hom}(\pi_1(X), \mathbb{R}(X)) \rightarrow \text{Hom}(\pi_1(Y), \mathbb{R}(Y)).$$

Then, for any $\omega \in Z_{\underline{\square}}^1(X)$ and $[\ell] \in \pi_1(X)$, it follows that

$$\rho(f^*(\omega))([\ell]) = \int_I (f^* \omega_{\underline{1}})(\ell) = \int_I \omega_{\underline{1}}(f \circ \ell) = \rho(\omega)([f \circ \ell]) = \rho(\omega) \circ f_*([\ell])$$

and hence we have $\rho \circ f^* = \text{Hom}(f_*, \text{id}) \circ \rho$ which implies the naturality of ρ .

Definition 6.5. For any differentiable space X , we define a groupoid \underline{X} in which the set of objects is equal to $X = \mathcal{O}(\pi_1(X))$, and the set of morphisms is obtained from $\mathcal{M}(\pi_1(X))$ by identifying all the morphisms which have starting and ending objects in common.

Then there clearly is a natural projection $\text{pr} : \pi_1(X) \rightarrow \underline{X}$ inducing a monomorphism $\text{pr}^* : \text{Hom}(\underline{X}, \mathbb{R}(X)) \hookrightarrow \text{Hom}(\pi_1(X), \mathbb{R}(X))$.

Definition 6.6. We denote the cokernel of pr^* by $\text{Hom}(\pi_1(X), \mathbb{R})$.

If $\omega = d\phi$ for some $\phi \in \Omega_{\underline{\square}}^0(X)$, then, for any path ℓ from x_0 to x_1 , we have $\rho(\omega)([\ell]) = \rho(d\phi)([\ell]) = \int_I d(\phi_I)(\ell) = [\phi_I(\ell)(t)]_{t=0}^{t=1} = \phi_I(\ell)(1) - \phi_I(\ell)(0)$, by the fundamental theorem of calculus. Hence $\phi_I(\ell)(\epsilon) = \phi_I(\ell)(\partial_1^\epsilon(*)) = \partial_1^{\epsilon*}(\phi_I(\ell))(*) = \phi_{\{*\}}(\ell \circ \partial_1^\epsilon)(*) = \phi_{\{*\}}(\ell(\epsilon))(*) = \phi_{\{*\}}(c_{x_\epsilon})(*)$ is depending only on x_ϵ the starting and ending objects of $[\ell] \in \pi_1(X)$. Thus the functor $\rho(\omega) : \pi_1(X) \rightarrow \mathbb{R}(X)$ induces a functor $\Phi(\omega) : \underline{X} \rightarrow \mathbb{R}(X)$ such that $\rho(\omega) = \Phi(\omega) \circ \text{pr}$, in other words, $\rho(B_{\underline{\square}}^1(X))$ is in the image of pr^* . Thus ρ induces a homomorphism $\rho_* : H_{\underline{\square}}^*(X) \rightarrow \text{Hom}(\pi_1(X), \mathbb{R})$.

Theorem 6.7. $\rho_* : H_{\underline{\square}}^1(X) \rightarrow \text{Hom}(\pi_1(X), \mathbb{R})$ is a monomorphism.

Proof: Assume that $\rho_*([\omega]) = 0$. Then we have $\rho(\omega) \in \text{Im pr}^*$. Thus there is a functor $\Phi(\omega) : \underline{X} \rightarrow \mathbb{R}$ such that $\rho(\omega) = \Phi(\omega) \circ \text{pr}$. Let $\{x_\alpha; \alpha \in \pi_0(X)\}$ be a complete set of representatives of $\pi_0(X)$. For any $P \in \mathcal{E}(\square^n)$, a map $F(P) : \square^n \rightarrow \mathbb{R}$ is given by

$$F(P)(\mathbf{x}) = \int_I \omega_{\underline{1}}(\ell_x) + \int_I \gamma_{\mathbf{x}}^* \omega_{\underline{n}}(\square^n), \quad x = P(\mathbf{0}),$$

where ℓ_x is a path from x_α , $\alpha = [x] \in \pi_0(X)$, to x in X and γ is a path from $\mathbf{0}$ to \mathbf{x} in \square^n . Then $F(P) : \square^n \rightarrow \wedge^0$ is well-defined smooth map by the equality $\int_I \omega_{\underline{1}}(\ell_x) = \rho(\omega)([\ell_x]) = \Phi(\omega)(\text{pr}([\ell_x]))$ which is not depending on the choice of ℓ_x , and hence it gives a 0-form $F : \mathcal{E}(\square^n) \rightarrow \wedge^0(\square^n)$ so that $dF = \omega$. Thus $[\omega] = 0$ and ρ_* is a monomorphism. \square

7. PARTITION OF UNITY

Let X be a differentiable space. In this section, we assume that there are subsets $A, B \subset X$ such that $\mathcal{U} = \{\text{Int } A, \text{Int } B\}$ gives an open covering of X .

Definition 7.1. A pair (ρ^A, ρ^B) of differentiable 0-forms ρ^A and ρ^B is called a partition of unity belonging to an open covering \mathcal{U} of X , if, for any plot $P : \square^n \rightarrow X$, $\text{Supp } \rho_{\underline{n}}^A(P) \subset P^{-1}(\text{Int } A)$, $\text{Supp } \rho_{\underline{n}}^B(P) \subset P^{-1}(\text{Int } B)$ and $\rho_{\underline{n}}^A(P) + \rho_{\underline{n}}^B(P) = 1$ on \square^n .

To obtain a well-defined smooth function by extending or gluing smooth functions on cubic sets, we use a fixed smooth stabilizer function $\hat{\lambda} : \mathbb{R} \rightarrow I$ (see [7]) which satisfies

$$(1) \quad \hat{\lambda}(-t) = 0, \quad \hat{\lambda}(1+t) = 1, \quad t \geq 0 \quad \text{and} \quad (2) \quad \hat{\lambda} \text{ is strictly increasing on } I = [0, 1].$$

Using $\hat{\lambda}$, we define a smooth function $\lambda_{a,b} : I \rightarrow I$, for any $a, b \in \mathbb{R}$ with $a < b$, by

$$\lambda_{a,b}(t) = \hat{\lambda}\left(\frac{t-a-\epsilon}{b-a-2\epsilon}\right)$$

for a small $\epsilon > 0$ enough to satisfy $\frac{b-a}{2} > \epsilon > 0$.

Using it, we show the existence of a partition of unity as follows.

Theorem 7.2. Let X be a differentiable space with an open covering $\{\text{Int } A, \text{Int } B\}$, $A, B \subset X$. Then there exists a partition of unity $\rho = \{\rho^A, \rho^B\}$ belonging to $\{\text{Int } A, \text{Int } B\}$. If the underlying topology on X is normal, ρ can be chosen as normal, in other words, there are closed sets G_A, G_B in X such that $X \setminus \text{Int } B \subset G_A \subset \text{Int } A$, $X \setminus \text{Int } A \subset G_B \subset \text{Int } B$ and $\text{Supp } \rho_{\underline{n}}^A(P) \subset P^{-1}(G_A)$ and $\text{Supp } \rho_{\underline{n}}^B(P) \subset P^{-1}(G_B)$ for all $n \geq 0$ and $P \in \mathcal{E}^X(\square^n)$.

The above theorem implies the exactness of Mayer-Vietoris exact sequence as follows.

Corollary 7.3. Let X be a differentiable space with an open covering $\mathcal{U} = \{\text{Int } A, \text{Int } B\}$, $A, B \subset X$. Then we have the following long exact sequence.

$$\begin{aligned} \cdots \rightarrow H_{\square}^q(X) &\rightarrow H_{\square}^q(A) \oplus H_{\square}^q(B) \rightarrow H_{\square}^q(A \cap B) \\ &\rightarrow H_{\square}^{q+1}(X) \rightarrow H_{\square}^{q+1}(A) \oplus H_{\square}^{q+1}(B) \rightarrow H_{\square}^{q+1}(A \cap B) \cdots \end{aligned}$$

Proof of Theorem 7.2. If X is normal, there is a continuous function $\rho : X \rightarrow I$ with $X \setminus \text{Int } B \subset \rho^{-1}(0)$ and $X \setminus \text{Int } A \subset \rho^{-1}(1)$. Otherwise, we define a function $\rho : X \rightarrow I$ by

$$\rho(x) = \begin{cases} 1, & x \in \text{Int } A \setminus \text{Int } B, \\ 1/2, & x \in \text{Int } A \cap \text{Int } B, \\ 0 & x \in \text{Int } B \setminus \text{Int } A. \end{cases}$$

Let $G_A = \rho^{-1}([0, \frac{2}{3}]) \subset X \setminus \rho^{-1}(1) \subset \text{Int } A$ and $G_B = \rho^{-1}([\frac{1}{3}, 1]) \subset X \setminus \rho^{-1}(0) \subset \text{Int } B$. Then $\text{Int } G_A \cup \text{Int } G_B \supset \rho^{-1}([0, \frac{2}{3})) \cup \rho^{-1}((\frac{1}{3}, 1]) = \rho^{-1}([0, \frac{2}{3}) \cup (\frac{1}{3}, 1]) = X$. Thus it is sufficient to construct a partition of unity $\{\rho^A, \rho^B\}$ belonging to $\mathcal{U} = \{\text{Int } G_A, \text{Int } G_B\}$: by induction on n , we construct functions $\rho_{\underline{n}}^A(P), \rho_{\underline{n}}^B(P) : \square^n \rightarrow I$ for any n -plot $P : \square^n \rightarrow X$, with conditions (1) through (4) below for $F = A, B$ and $\epsilon = 0, 1$.

- (1) a) $\rho_{\underline{n}}^F(P \circ \varepsilon_i) = \rho_{\underline{n-1}}^F(P) \circ \varepsilon_i$, $1 \leq i \leq n+1$, b) $\rho_{\underline{n-1}}^F(P \circ \partial_i^\epsilon) = \rho_{\underline{n}}^F(P) \circ \partial_i^\epsilon$, $1 \leq i \leq n$,
- (2) $\rho_{\underline{n}}^A(P) + \rho_{\underline{n}}^B(P) = 1 : \square^n \rightarrow \mathbb{R}$, (3) $\text{Supp } \rho_{\underline{n}}^F(P) \subset P^{-1}(\text{Int } G_F) \subset \square^n$,
- (4) $\rho_F(P) \circ \partial_i^{1-t} = \rho_F(P) \circ \partial_i^1$ and $\rho_F(P) \circ \partial_i^t = \rho_F(P) \circ \partial_i^0$ for all $0 \leq t \leq a$ for sufficiently small $a > 0$, where ∂_i^t is defined by $\partial_i^t(t_1, \dots, t_{n-1}) = (t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_{n-1})$.

($n = 0$) For any plot $P : \square^0 = \{*\} \rightarrow X$, we define $\rho_{\underline{n}}^A(P) = \rho(P(*))$ and $\rho_{\underline{n}}^B(P) = 1 - \rho_{\underline{n}}^A(P)$, which satisfy (2) and (3), though (1) and (4) are empty conditions in this case.

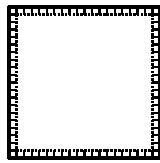
($n > 0$) We may assume a plot $P : \square^n \rightarrow X$ is non-degenerate by (1) a).

Firstly, $P^{-1}\mathcal{U} = \{P^{-1}(\text{Int } A), P^{-1}(\text{Int } B)\}$ is an open covering of $\square^n \subset \mathbb{R}^n$, and hence we have a partition of unity $\{\varphi^A, \varphi^B\}$ belonging to $P^{-1}\mathcal{U}$ on \square^n .

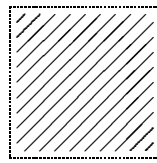
Secondly, by the induction hypothesis, there is a small $a > 0$ for the condition (4). Let U_a be the a -neighbourhood of $\partial \square^n$. For $F = A, B$, we define $\hat{\rho}_{\underline{n}}^F(P) : U_a \rightarrow \mathbb{R}$ by

$$\hat{\rho}_{\underline{n}}^F(P) \circ \partial_i^{\epsilon \pm t} = \rho_{\underline{n-1}}^F(P \circ \partial_i^\epsilon), \quad 0 \leq t < a, \quad 1 \leq i \leq n, \quad \epsilon = 0, 1,$$

where we denote $\epsilon \pm t = \epsilon + (-1)^\epsilon t$, and then we obtain $\text{Supp } \hat{\rho}_{\underline{n}}^F(P) \subset P^{-1}(\text{Int } G_F) \cap U_a$, if we choose $a > 0$ small enough.



U_a



$\text{Int } \square^n$

Thirdly, since two open sets U_a and $\text{Int } \square^n$ form an open covering of \square^n , we also have a partition of unity $(\psi_\partial, \psi_\circ)$ belonging to $\{U_a, \text{Int } \square^n\}$ given by $\psi_\partial = (\lambda_{1-a,1})^n$ and $\psi_\circ = 1 - \psi_\partial$ so that we have $\text{Supp } \psi_\partial \subset U_a$ and $\text{Supp } \psi_\circ \subset \text{Int } \square^n$. Then, for $F = A, B$, $\psi_\partial|_{U_a} \cdot \hat{\rho}_n^F(P)$ is defined on U_a with value 0 on $U_a \setminus \text{Supp } \psi_\partial$. Hence by filling 0 outside $\text{Supp } \psi_\partial$, we obtain a smooth map $\widehat{\psi_\partial \rho_n^F} : \square^n \rightarrow \mathbb{R}$ on entire \square^n , as the 0-extension of $\psi_\partial|_{U_a} \cdot \hat{\rho}_n^F(P) : U_a \rightarrow \mathbb{R}$.

Finally, let $\rho_n^F(P) = \widehat{\psi_\partial \rho_n^F} + \psi_\circ \cdot \varphi^F$ for $F = A, B$. Then $\text{Supp } \rho_n^F(P) \subset \text{Supp } \widehat{\psi_\partial \rho_n^F} \cup \text{Supp}(\psi_\circ \cdot \varphi^F) \subset (\text{Supp } \psi_\partial \cap \text{Supp } \hat{\rho}_n^F) \cup (\text{Supp } \psi_\circ \cap \text{Supp } \varphi^F) \subset (U_a \cap P^{-1}(\text{Int } G_F)) \cup (\text{Int } \square^n \cap P^{-1}(\text{Int } G_F)) = P^{-1}(\text{Int } G_F)$. By definition, we also have

$$\rho_n^A(P) + \rho_n^B(P) = \widehat{\psi_\partial \rho_n^A} + \widehat{\psi_\partial \rho_n^B} + \psi_\circ \cdot \varphi^A + \psi_\circ \cdot \varphi^B = \psi_\partial + \psi_\circ = 1 \quad \text{on } \square^n,$$

which implies that $(\rho_n^A(P), \rho_n^B(P))$ gives a partition of unity belonging to the open covering $\{P^{-1}(\text{Int } A), P^{-1}(\text{Int } B)\}$ of \square^n . By definition, $(\rho_n^A(P), \rho_n^B(P))$ satisfies the conditions (1) through (4), and it completes the induction step. The latter part is clear. \square

8. EXCISION THEOREM

Let $X = (X, \mathcal{E}^X)$ be a differentiable space and \mathcal{U} an open covering of X . We denote $\mathcal{E}^{\mathcal{U}} = \{P \in \mathcal{E}^X; \text{Im } P \subset U \text{ for some } U \in \mathcal{U}\}$. Then we regard $\mathcal{E}^{\mathcal{U}}$ as a functor $\mathcal{E}^{\mathcal{U}} : \text{Convex} \rightarrow \text{Set}$ which is given by $\mathcal{E}^{\mathcal{U}}(C) = \{P \in \mathcal{E}^{\mathcal{U}}, \text{Dom } P = C\}$ for $C \in \mathcal{O}(\text{Convex})$ and $\mathcal{E}^{\mathcal{U}}(f) = \mathcal{E}^X(f)|_{\mathcal{E}^{\mathcal{U}}(C)} : \mathcal{E}^{\mathcal{U}}(C) \rightarrow \mathcal{E}^{\mathcal{U}}(C')$ for a smooth map $f : C' \rightarrow C$ in Convex . When $\mathcal{U} = \{X\}$, we have $\mathcal{E}^{\{X\}} = \mathcal{E}^X$. We also denote $\mathcal{E}_{\square}^{\mathcal{U}} = \mathcal{E}^{\mathcal{U}} \circ \square : \square \rightarrow \text{Set}$.

Definition 8.1. A natural transformation $\omega : \mathcal{E}_{\square}^{\mathcal{U}} \rightarrow \wedge_{\square}^p$ is called a cubical differential p -form w.r.t. an open covering \mathcal{U} of X . $\Omega_{\square}^p(\mathcal{U})$ denotes the set of all cubical differential p -form w.r.t. an open covering \mathcal{U} of X . For example, $\Omega_{\square}^p(\{X\}) = \Omega_{\square}^p(X)$.

We introduce a notion of a q -cubic set in \mathbb{R}^n using induction on $q \geq -1$ up to n .

$(q = -1)$: The empty set \emptyset is a -1 -cubic set in \mathbb{R}^n .

$(n \geq q \geq 0)$: (1) if $\sigma \subset L$ is a $(q-1)$ -cubic set in \mathbb{R}^n and $\mathbf{b} \notin L$, where L is a hyperplane of dimension $q-1$ in \mathbb{R}^n , then $\sigma * \mathbf{b} = \{t\mathbf{x} + (1-t)\mathbf{b}; \mathbf{x} \in \sigma, t \in I\}$ is a q -cubic set in \mathbb{R}^n with faces τ and $\tau * \mathbf{b}$, where τ is a face of σ , including \emptyset and $\emptyset * \mathbf{b} = \mathbf{b}$.

(2) if $\sigma \subset \mathbb{R}^{i-1} \times \{0\} \times \mathbb{R}^{n-i}$ is a $(q-1)$ -cubic set in \mathbb{R}^n with $q \geq 1$, then the product set $\sigma \times I = \{(\mathbf{x}_{i-1}, t, \mathbf{x}'_{n-i}); (\mathbf{x}_{i-1}, 0, \mathbf{x}'_{n-i}) \in \sigma, t \in I\}$ is a q -cubic set in \mathbb{R}^n with faces $\tau \times \{0\}$, $\tau \times \{1\}$ and $\tau \times I$, where τ is a face of σ , including \emptyset .

We denote by $C(n)^q$ the set of q -cubic sets in \mathbb{R}^n and $C(n) = \{\emptyset\} \cup \bigcup_{q \geq 0} C(n)^q$, $n \geq 0$. We denote $\tau < \sigma$ if $\tau \in C(n)$ is a face of $\sigma \in C(n)$ and denote $\partial\sigma = \bigcup_{\tau < \sigma} \tau$. We fix a relative diffeomorphism $\phi_\sigma : (\square^q, \partial\square^q) \rightarrow (\sigma, \partial\sigma)$ for each q -cubic set σ in \mathbb{R}^n , $q \geq 0$.

A subset $K \subset C(n)$ is called a cubical complex if it satisfies the following conditions.

- (1) $\emptyset \in K$, (2) $\tau < \sigma \ \& \ \sigma \in K \implies \tau \in K$,
- (3) $\tau, \sigma \in K \implies \tau \cap \sigma \in K \ \& \ \tau \cap \sigma < \tau \ \& \ \tau \cap \sigma < \sigma$.

For any cubical complex $K \subset C(n)$, we denote $K^q = \{\sigma \in K; \sigma \text{ is a } q\text{-cubic set}\}$, $n \geq 0$ and $|K| = \bigcup_{\sigma \in K} \sigma$. For any cubical complexes K and L , a map $f : |L| \rightarrow |K|$ in **Convex** is called ‘polyhedral’ w.r.t. L and K , if $f(\sigma) \in K$ for any $\sigma \in L$. If a cubical complex $K \subset C(n)$ satisfies $|K| = \square^n$, we call K a ‘cubical subdivision’ of an n -cube \square^n .

Definition 8.2. We define a category $\text{SubDiv}_{\mathcal{U}}$ as follows:

- Object:** $\mathcal{O}(\text{SubDiv}_{\mathcal{U}}) = \{(K, P) \in C(n) \times \mathcal{E}^X(\square^n); |K| = \square^n, \forall_{\sigma \in K} P|_{\sigma} \in \mathcal{E}^{\mathcal{U}}, n \geq 0\}$,
- Morphism:** $\text{SubDiv}_{\mathcal{U}}((L, Q), (K, P)) = \{f : |L| \subset |K| \text{ polyhedral}; Q = P|_{|L|}\}$.

Let $\text{SubDiv}_X = \text{SubDiv}_{\{X\}}$. Then there is an embedding $\iota_{SD}^{\mathcal{U}} : \text{SubDiv}_{\mathcal{U}} \hookrightarrow \text{SubDiv}_X$.

Theorem 8.3. There is a functor $\text{Sd}_{\mathcal{U}}^* : \text{SubDiv}_X \rightarrow \text{SubDiv}_{\mathcal{U}}$ such that $\text{Sd}_{\mathcal{U}}^* \circ \iota_{SD}^{\mathcal{U}} = \text{id}$.

Proof: We construct a functor $\text{Sd}_{\mathcal{U}} : \text{SubDiv}_X \rightarrow \text{SubDiv}_X$ satisfying $\text{Sd}_{\mathcal{U}} \circ \iota_{SD}^{\mathcal{U}} = \iota_{SD}^{\mathcal{U}}$.

Firstly, for $(K, P) \in \mathcal{O}(\text{SubDiv}_X)$, $K \subset C(n)$ is a cubical subdivision of $|K| = \square^n = \text{Dom } P$. Let $K_P(\mathcal{U}) = \{\sigma \in K; \exists U \in \mathcal{U} P(\sigma) \subset U\} < K$. We define $\text{Sd}_{\mathcal{U}}(K, P) = (\text{Sd}_{\mathcal{U}}^{\mathcal{U}}(K), P)$ by induction on dimension of a cubic set in K .

$$\text{Sd}_{\mathcal{U}}^{\mathcal{U}}(K)^0 = K^0 \cup \{\mathbf{b}_{\sigma}; \sigma \in K \setminus K_P(\mathcal{U})\},$$

$$\text{Sd}_{\mathcal{U}}^{\mathcal{U}}(K)^q = K_P(\mathcal{U})^q \cup \{\rho * \mathbf{b}_{\sigma}; \rho \in \text{Sd}_{\mathcal{U}}^{\mathcal{U}}(\partial\sigma)^{q-1}, \sigma \in K \setminus K_P(\mathcal{U})\},$$

where $\partial\sigma$ denotes the subcomplex $\{\tau \in K; \tau < \sigma\}$ of K .

Secondly, for any map $f : (L, Q) \rightarrow (K, P)$, we have $L \subset K$ and $Q = P|_{|L|}$. Then by definition, we have $\text{Sd}_{\mathcal{U}}(L) \subset \text{Sd}_{\mathcal{U}}(K)$, and hence the inclusion $f : |\text{Sd}_{\mathcal{U}}(L)| = |L| \subset |K| = |\text{Sd}_{\mathcal{U}}(K)|$ is again polyhedral. Thus we obtain $\text{Sd}_{\mathcal{U}}(f) = f : \text{Sd}_{\mathcal{U}}(L, Q) \rightarrow \text{Sd}_{\mathcal{U}}(K, P)$.

Thirdly, we give a distance of subcomplexes K and $K_P(\mathcal{U})$ defined as follows:

$$\varepsilon_P^{\mathcal{U}}(K) = \text{Min} \{d(\tau, \mathbf{x}) \mid \tau \subset P^{-1}(U) \not\supset \mathbf{x}, U \in \mathcal{U} \ \& \ \tau \text{ is maximal in } K_P(\mathcal{U})\},$$

$$d_P^{\mathcal{U}}(K) = \text{Max} \{d(\tau, \mathbf{x}) \mid \tau \cap \sigma \neq \emptyset, \mathbf{x} \in \sigma \in K \ \& \ \tau \text{ is maximal in } K_P(\mathcal{U})\},$$

where $d(\tau, \mathbf{x})$ denotes the distance in \square^n of τ and \mathbf{x} , and hence $\varepsilon_P^{\mathcal{U}}(K) > 0$. We can easily see that $d_{\mathcal{U}}(\text{Sd}_P^{\mathcal{U}}(K)) \leq \frac{n}{n+1} d_{\mathcal{U}}(K)$ and hence that, for sufficiently large $r > 0$, the r -times iteration of $\text{Sd}_P^{\mathcal{U}}$ satisfies $d_P^{\mathcal{U}}((\text{Sd}_P^{\mathcal{U}})^r(K)) < \varepsilon_P^{\mathcal{U}}(K)$. Thus $\text{Sd}_{\mathcal{U}}^r(K, P) \in \text{SubDiv}_{\mathcal{U}}$.

Finally, when $(K, P) \in \mathbf{SubDiv}_{\mathcal{U}}$, we have $\mathrm{Sd}_P^{\mathcal{U}}(K, P) = (K, P)$ by definition, and hence $\mathrm{Sd}_{\mathcal{U}}^*$ the sufficiently many times iteration of $\mathrm{Sd}_{\mathcal{U}}$ on each (K, P) is a desired functor. \square

Definition 8.4. A functor $\mathrm{Td}_{\mathcal{U}} : \mathbf{SubDiv}_X \rightarrow \mathbf{SubDiv}_X$ given by $\mathrm{Td}_{\mathcal{U}}(K, P) = (\mathrm{Td}_P^{\mathcal{U}}(K), \hat{P})$ for $(K, P) \in \mathcal{O}(\mathbf{SubDiv}_X)$ is defined as follows: we denote $\hat{P} = P \circ \mathrm{pr}_1 : \square^n \times I \rightarrow X$ which is a plot in $\mathcal{E}^X(\square^{n+1})$. Then a cubical subdivision $\mathrm{Td}_P^{\mathcal{U}}(K)$ of \square^{n+1} is defined as follows:

$$\begin{aligned} \mathrm{Td}_P^{\mathcal{U}}(K)^0 &= K^0 \times \{0\} \cup \mathrm{Sd}_P^{\mathcal{U}}(K)^0 \times \{1\}, \\ \mathrm{Td}_P^{\mathcal{U}}(K)^q &= K^q \times \{0\} \cup \mathrm{Sd}_P^{\mathcal{U}}(K)^q \times \{1\} \cup K_P(\mathcal{U})^{q-1} \times I \\ &\quad \cup \{\rho * (\mathbf{b}_{\sigma}, 1); \rho \in \mathrm{Td}_P^{\mathcal{U}}(\partial\sigma)^{q-1}, \sigma \in K \setminus K_P(\mathcal{U})\}. \end{aligned}$$

Also for a map $f : (L, Q) \rightarrow (K, P)$, we have $L \subset K$ and $Q = P|_{|L|}$. Then by definition, we have $\mathrm{Td}_{\mathcal{U}}(L) \subset \mathrm{Td}_{\mathcal{U}}(K)$, and hence the inclusion $f \times \mathrm{id} : |\mathrm{Td}_{\mathcal{U}}(L)| = |L| \times I \subset |K| \times I = |\mathrm{Td}_{\mathcal{U}}(K)|$ is again polyhedral. Thus we obtain $\mathrm{Td}_{\mathcal{U}}(f) = f : \mathrm{Td}_{\mathcal{U}}(L, Q) \rightarrow \mathrm{Td}_{\mathcal{U}}(K, P)$.

Definition 8.5. For any cubical differential p -form $\omega \in \Omega_{\square}^p(\mathcal{U})$, we have a cubical differential p -form $\tilde{\omega} \in \Omega_{\square}^p(\mathcal{U})$ defined by $\tilde{\omega}_{\underline{n}}(P) = (\lambda^n)^* \omega_{\underline{n}}(P)$ for any $P \in \mathcal{E}_{\underline{n}}^{\mathcal{U}}$, $\lambda = \lambda_{0,1}$. In addition, if ω is a differential p -form with compact support, then so is $\tilde{\omega}$.

Lemma 8.6. There is a homomorphism $D_{\mathcal{U}} : \Omega_{\square}^*(\mathcal{U}) \rightarrow \Omega_{\square}^*(\mathcal{U})$ such that $dD_{\mathcal{U}}(\omega)_{\underline{n}} + D_{\mathcal{U}}(d\omega)_{\underline{n}} = \tilde{\omega}_{\underline{n}} - \omega_{\underline{n}}$ and $D_{\mathcal{U}}(\Omega_{\square_c}^p(\mathcal{U})) \subset \Omega_{\square_c}^{p-1}(\mathcal{U})$ for any $p \geq 0$.

Proof: Let $H : I \times I \rightarrow I$ be a smooth homotopy between $\mathrm{id} : I \rightarrow I$ and $\lambda : I \rightarrow I$, which gives rise to a smooth homotopy $H_n : \square^{n+1} = I \times \square^n \rightarrow \square^n$ of $\mathrm{id} : \square^n \rightarrow \square^n$ and $\lambda^n : \square^n \rightarrow \square^n$, $n \geq 0$. Then we have $H_n \circ \mathrm{in}_0 = \mathrm{id}$ and $H_n \circ \mathrm{in}_1 = \lambda^n$, where $\mathrm{in}_t : \square^n \hookrightarrow I \times \square^n$ is given by $\mathrm{in}_t(\mathbf{x}) = (t, \mathbf{x})$. For any cubical differential p -form $\omega : \mathcal{E}_{\square}^{\mathcal{U}} \rightarrow \wedge_{\square}^p$, a cubical $(p-1)$ -form $D_{\mathcal{U}}(\omega) : \mathcal{E}_{\square}^{\mathcal{U}} \rightarrow \wedge_{\square}^{p-1}$ is defined on a plot $P \in \mathcal{E}_{\square}^{\mathcal{U}}$, by the following formula.

$$\begin{aligned} D_{\mathcal{U}}(\omega)_{\underline{n}}(P) &= \int_I H^* \omega_{\underline{n}}(P) : \square^n \rightarrow \wedge^{p-1}(T_n^*), \\ \left[\int_I H^* \omega_{\underline{n}}(P) \right](\mathbf{x}) &= \sum_{i_2, \dots, i_p} \int_0^1 a_{i_2, \dots, i_p}(t, \mathbf{x}) dt \cdot dx_{i_2} \wedge \cdots \wedge dx_{i_p}, \end{aligned}$$

where we assume $H^* \omega_{\underline{n}}(P) = \sum_{i_2, \dots, i_p} a_{i_2, \dots, i_p}(t, \mathbf{x}) dt \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} + \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(t, \mathbf{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_p} : I \times \square^n \rightarrow \wedge^{p-1}(T_{n+1}^*)$, $(t, \mathbf{x}) \in I \times \square^n$ and $T_{n+1}^* = \mathbb{R} dt \oplus \bigoplus_{i=1}^n \mathbb{R} dx_i$.

First, let $\mathrm{in}_t : \square^n \rightarrow I \times \square^n$ be the inclusion defined by $\mathrm{in}_t(\mathbf{x}) = (t, \mathbf{x})$ for $t = 0, 1$. By $H \circ \mathrm{in}_0 = \mathrm{id}$, we have $\omega_{\underline{n}}(P) = \mathrm{id}^* \omega_{\underline{n}}(P) = \mathrm{in}_0^* H^* \omega_{\underline{n}}(P) = \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(0, \mathbf{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. On the other hand by $H \circ \mathrm{in}_1 = \lambda^n$, we have $(\lambda^n)^* \omega_{\underline{n}}(P) = \mathrm{in}_1^* H^* \omega_{\underline{n}}(P) = \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(1, \mathbf{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ for any $\mathbf{x} \in \square^n$.

Second, by definition, we have $dH^*\omega_{\underline{n}}(P) = \sum_i \sum_{i_2, \dots, i_p} \frac{\partial a_{i_2, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dx_i \wedge dt \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p} + \sum_{i_1, \dots, i_p} \frac{\partial b_{i_1, \dots, i_p}}{\partial t}(t, \mathbf{x}) dt \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p} + \sum_i \sum_{i_2, \dots, i_p} \frac{\partial b_{i_1, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dx_i \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}$, and hence we obtain $D_{\mathcal{U}}(d\omega)_{\underline{n}}(P) = \int_I H^* d\omega_{\underline{n}}(P) = -\sum_i \sum_{i_2, \dots, i_p} \int_I \frac{\partial a_{i_2, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dt \cdot dx_i \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p} + \sum_{i_1, \dots, i_p} \int_I \frac{\partial b_{i_1, \dots, i_p}}{\partial t}(t, \mathbf{x}) dt \cdot dx_{i_1} \wedge \dots \wedge dx_{i_p}$, $(t, \mathbf{x}) \in I \times \square^n$.

Third, we have $D_{\mathcal{U}}(\omega)_{\underline{n}}(P) = \sum_{i_2, \dots, i_p} \int_I a_{i_2, \dots, i_p}(t, \mathbf{x}) dt \cdot dx_{i_2} \wedge \dots \wedge dx_{i_p}$, and hence we obtain $dD_{\mathcal{U}}(\omega)_{\underline{n}}(P) = \sum_i \sum_{i_2, \dots, i_p} \int_I \frac{\partial a_{i_2, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dt \cdot dx_i \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p}$, $(t, \mathbf{x}) \in I \times \square^n$.

Hence $[dD_{\mathcal{U}}(\omega)_{\underline{n}}(P) + D_{\mathcal{U}}(d\omega)_{\underline{n}}(P)](\mathbf{x}) = \sum_{i_1, \dots, i_p} \int_I \frac{\partial b_{i_1, \dots, i_p}}{\partial t}(t, \mathbf{x}) dt \cdot dx_{i_1} \wedge \dots \wedge dx_{i_p} = \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(1, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p} - \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(0, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p}$, $\mathbf{x} \in \square^n$. Thus we obtain $dD_{\mathcal{U}}(\omega)(P) + D_{\mathcal{U}}(d\omega)(P) = \tilde{\omega}(P) - \omega(P)$. By the above construction of $D_{\mathcal{U}}$, it is clear to see $D_{\mathcal{U}}(\Omega_{\square_c}^p(\mathcal{U})) \subset \Omega_{\square_c}^{p-1}(\mathcal{U})$, and it completes the proof of the lemma. \square

Remark 8.7. We have $b_{i_1, \dots, i_p}(1, \mathbf{x}) = b_{i_1, \dots, i_p}(0, \lambda^n(\mathbf{x}))\lambda'(x_{i_1}) \cdots \lambda'(x_{i_p})$ for $1 \leq i_1 < \dots < i_p \leq n$ and $\mathbf{x} = (x_1, \dots, x_n) \in \square^n$, since $(\lambda^n)^*\omega_{\underline{n}}(P) = in_1^*H^*\omega_{\underline{n}}(P)$.

Let $\omega \in \Omega_{\square}^*(X)$ and $P \in \mathcal{E}^X(\square^n)$. Then a cubical complex $K = \{\sigma; \sigma < \square^n\}$ derives cubical subdivisions $K_r = (\text{Sd}_P^{\mathcal{U}})^r(K)$ and $K_* = (\text{Sd}_P^{\mathcal{U}})^*(K)$ where $K_* = K_r$ for sufficiently large $r \geq 0$. We define $\omega^{(r)} \in \Omega_{\square}^p(\mathcal{U})$, $r \geq 0$, as follows: for any $\sigma \in K_r$,

$$\omega_{\underline{n}}^{(r)}(P)|_{\text{Int } \sigma} = \hat{\omega}_{\sigma}^{(r)}(P|_{\sigma})|_{\text{Int } \sigma},$$

where $\hat{\omega}_{\sigma}^{(r)}(P|_{\sigma})|_{\text{Int } \sigma} = \omega_{\underline{n}}(P|_{\sigma} \circ \phi_{\sigma}) \circ \lambda^n \circ \phi_{\sigma}^{-1} : \text{Int } \sigma \xrightarrow{\phi_{\sigma}^{-1}} \text{Int } \square^n \xrightarrow{\lambda^n} \text{Int } \square^n \xrightarrow{\omega_{\underline{n}}(P \circ \phi_{\sigma})} \wedge^p$. Then by definition, $\omega_{\underline{n}}^{(r)}(P)|_{\text{Int } \sigma}$ can be smoothly extended to $\partial\sigma$, and hence $\omega_{\underline{n}}^{(r)}(P) : \square^n \rightarrow \wedge_{T_n^*}^p$ is well-defined and we obtain $\omega^{(r)} \in \Omega_{\square}^p(X)$.

Lemma 8.8. There is a homomorphism $D_{\mathcal{U}}^{(r)} : \Omega_{\square}^*(X) \rightarrow \Omega_{\square}^*(X)$ such that $dD_{\mathcal{U}}^{(r)}(\omega) + D_{\mathcal{U}}^{(r)}(d\omega) = \omega^{(r+1)} - \omega^{(r)}$ and $D_{\mathcal{U}}^{(r)}(\Omega_{\square_c}^p(\mathcal{U})) \subset \Omega_{\square_c}^{p-1}(\mathcal{U})$ for $p \geq 0$.

Proof: For any $\omega \in \Omega_{\square}^p(\mathcal{U})$, we define $D_{\mathcal{U}}^{(r)}(\omega) \in \Omega_{\square}^p(X)$ as follows: let $P \in \mathcal{E}^X(\square^n)$. We have a cubical complex $K = \{\sigma; \sigma < \square^n\}$ which derives cubical subdivisions $K_r = (\text{Sd}_P^{\mathcal{U}})^r(K)$ of \square^n and $\hat{K}_r = \text{Td}_P^{\mathcal{U}}(K_r)$ of $I \times \square^n$ so that $\text{in}_0^* \hat{K}_r = K_r$ and $\text{in}_1^* \hat{K}_r = K_{r+1}$. Now we define a smooth function $\hat{\omega}(P) : I \times \square^n \rightarrow \wedge^p(T_{n+1}^*)$ as follows: for any $\sigma \in \hat{K}_r^{n+1}$,

$$\hat{\omega}(P)|_{\text{Int } \sigma} = \hat{\omega}'_{\sigma}(P \circ \text{pr}_2|_{\sigma})|_{\text{Int } \sigma} : I \times \square^n \longrightarrow \wedge^p(T_{n+1}^*),$$

where $\widehat{\omega}'_\sigma(P \circ \text{pr}_2|_\sigma)|_{\text{Int } \sigma} = \omega_{\underline{n+1}}(P \circ \text{pr}_2|_\sigma \circ \phi_\sigma) \circ \lambda^{n+1} \circ \phi_\sigma^{-1} : \text{Int } \sigma \xrightarrow{\phi_\sigma^{-1}} \text{Int } \square^{n+1} \xrightarrow{\lambda^{n+1}} \text{Int } \square^{n+1} \xrightarrow{\omega_{\underline{n+1}}(P \circ \text{pr}_2 \circ \phi_\sigma)} \wedge^p$. Then by definition, $\widehat{\omega}'_\sigma(P \circ \text{pr}_2|_\sigma)|_{\text{Int } \sigma}$ can be smoothly extended to σ and we obtain a smooth function $\widehat{\omega}(P) : I \times \square^n \rightarrow \wedge_{T_{n+1}^*}^p$.

First, a cubical $(p-1)$ -form $D_{\mathcal{U}}^{(r)}(\omega) \in \Omega_{\square}^{p-1}(X)$ is defined as follows: for any cubical differential p -form $\omega : \mathcal{E}_{\square}^X \rightarrow \wedge_{\square}^p$ on a plot $P \in \mathcal{E}_{\square}^X$,

$$D_{\mathcal{U}}^{(r)}(\omega)_{\underline{n}}(P) = \int_I \widehat{\omega}(P) : \square^n \rightarrow \wedge^{p-1}(T_n^*),$$

$$\left[\int_I \widehat{\omega}(P) \right] (\mathbf{x}) = \sum_{i_2, \dots, i_p} \int_0^1 a_{i_2, \dots, i_p}(t, \mathbf{x}) dt \cdot dx_{i_2} \wedge \dots \wedge dx_{i_p},$$

where $\widehat{\omega}(P) = \sum_{i_2, \dots, i_p} a_{i_2, \dots, i_p}(t, \mathbf{x}) dt \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p} + \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(t, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p} : I \times \square^n \rightarrow \wedge^{p-1}(T_{n+1}^*)$, $(t, \mathbf{x}) \in I \times \square^n$ and $T_{n+1}^* = \mathbb{R} dt \oplus \bigoplus_{i=1}^n \mathbb{R} dx_i$. Then, since $\text{in}_0^* \widehat{K}_r = K_r$ and $\text{in}_1^* \widehat{K}_r = K_{r+1}$, we easily see that $\omega_{\underline{n}}^{(r)}(P) = \text{in}_0^* \widehat{\omega}(P) = \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(0, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p}$ and $\omega_{\underline{n}}^{(r+1)}(P) = \text{in}_1^* \widehat{\omega}(P) = \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(1, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p}$.

Second, by definition, we have $\widehat{d\omega}(P) = d\widehat{\omega}(P) = \sum_i \sum_{i_2, \dots, i_p} \frac{\partial a_{i_2, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dx_i \wedge dt \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p} + \sum_{i_1, \dots, i_p} \frac{\partial b_{i_1, \dots, i_p}}{\partial t}(t, \mathbf{x}) dt \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p} + \sum_i \sum_{i_1, \dots, i_p} \frac{\partial b_{i_1, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dx_i \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}$, and hence $D_{\mathcal{U}}^{(r)}(d\omega)_{\underline{n}}(P) = \int_I \widehat{d\omega}(P) = - \sum_i \sum_{i_2, \dots, i_p} \int_I \frac{\partial a_{i_2, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dt \cdot dx_i \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p} + \sum_{i_1, \dots, i_p} \int_I \frac{\partial b_{i_1, \dots, i_p}}{\partial t}(t, \mathbf{x}) dt \cdot dx_{i_1} \wedge \dots \wedge dx_{i_p}$, $(t, \mathbf{x}) \in I \times \square^n$.

Third, we have $D_{\mathcal{U}}^{(r)}(\omega)_{\underline{n}}(P) = \sum_{i_2, \dots, i_p} \int_I a_{i_2, \dots, i_p}(t, \mathbf{x}) dt \cdot dx_{i_2} \wedge \dots \wedge dx_{i_p}$, and hence we obtain $dD_{\mathcal{U}}^{(r)}(\omega)_{\underline{n}}(P) = \sum_i \sum_{i_2, \dots, i_p} \int_I \frac{\partial a_{i_2, \dots, i_p}}{\partial x_i}(t, \mathbf{x}) dt \cdot dx_i \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p}$, $(t, \mathbf{x}) \in I \times \square^n$.

Hence $\left[dD_{\mathcal{U}}^{(r)}(\omega)_{\underline{n}}(P) + D_{\mathcal{U}}^{(r)}(d\omega)_{\underline{n}}(P) \right] (\mathbf{x}) = \sum_{i_1, \dots, i_p} \int_I \frac{\partial b_{i_1, \dots, i_p}}{\partial t}(t, \mathbf{x}) dt \cdot dx_{i_1} \wedge \dots \wedge dx_{i_p} = \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(1, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p} - \sum_{i_1, \dots, i_p} b_{i_1, \dots, i_p}(0, \mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_p}$, $\mathbf{x} \in \square^n$. Thus we obtain $dD_{\mathcal{U}}^{(r)}(\omega)(P) + D_{\mathcal{U}}^{(r)}(d\omega)(P) = \omega^{(r+1)}(P) - \omega^{(r)}(P)$. By the above construction of $D_{\mathcal{U}}^{(r)}$, it is clear to see that $D_{\mathcal{U}}^{(r)}(\Omega_{\square_c}^p(\mathcal{U})) \subset \Omega_{\square_c}^{p-1}(\mathcal{U})$. \square

Theorem 8.9. *The restriction $\text{res} : \Omega_{\square}^*(X) \rightarrow \Omega_{\square}^*(\mathcal{U})$ induces an isomorphism of cubical de Rham cohomologies $\text{res}^* : H_{\square}^*(X) \rightarrow H_{\square}^*(\mathcal{U})$. In addition, res induces a map $\text{res} : \Omega_{\square_c}^*(X) \rightarrow \Omega_{\square_c}^*(\mathcal{U})$ which further induces an isomorphism $\text{res}^* : H_{\square_c}^*(X) \rightarrow H_{\square_c}^*(\mathcal{U})$.*

Proof: For any $\omega \in \Omega_{\square}^p(\mathcal{U})$, we define $\omega^* \in \Omega_{\square}^p(X)$ as follows: let $P \in \mathcal{E}^X(\square^n)$. Then we obtain a cubical complex $K = \{\sigma; \sigma < \square^n\}$ which derives a cubical subdivision $K_* = (\text{Sd}_P^{\mathcal{U}})^*(K)$. We define cubical differential p -forms $\omega^* \in \Omega_{\square}^p(\mathcal{U})$ as follows: for any $\sigma \in K_*$,

$$\omega_{\underline{n}}^*(P)|_{\text{Int } \sigma} = \hat{\omega}_{\sigma}^*(P|_{\sigma})|_{\text{Int } \sigma},$$

where $\hat{\omega}_{\sigma}^*(P|_{\sigma})|_{\text{Int } \sigma} = \omega_{\underline{n}}(P|_{\sigma \circ \phi_{\sigma}}) \circ \lambda^n \circ \phi_{\sigma}^{-1} : \text{Int } \sigma \xrightarrow{\phi_{\sigma}^{-1}} \text{Int } \square^n \xrightarrow{\lambda^n} \text{Int } \square^n \xrightarrow{\omega_{\underline{n}}(P \circ \phi_{\sigma})} \wedge^p$. Then by definition, $\omega_{\underline{n}}^*(P)|_{\text{Int } \sigma}$ can be uniquely extended to $\partial \sigma$ and we obtain $\omega_{\underline{n}}^*(P) : \square^n \rightarrow \wedge_{T_n}^p$ so that $\omega^* \in \Omega_{\square}^p(X)$ whose restriction to $\Omega_{\square}^p(\mathcal{U})$ equals, by definition, to $\tilde{\omega}$ with a $(p-1)$ -form $D_{\mathcal{U}}(\omega) \in \Omega_{\square}^{p-1}(\mathcal{U})$ satisfying $dD_{\mathcal{U}}(\omega) = \tilde{\omega} - \omega$ if $d\omega = 0$, by Lemma 8.6. If $d\omega = 0$, then $d\hat{\omega}^* = 0$, and hence $d\omega^* = 0$. Thus the restriction $\text{res} : \Omega_{\square}^*(X) \rightarrow \Omega_{\square}^*(\mathcal{U})$ induces an epimorphism $\text{res}^* : H_{\square}^*(X) \rightarrow H_{\square}^*(\mathcal{U})$ of cubical de Rham cohomologies.

So we are left to show that $\text{res}^* : H_{\square}^*(X) \rightarrow H_{\square}^*(\mathcal{U})$ is a monomorphism: let $\omega \in \Omega_{\square}^p(X)$. Then we obtain a cubical differential p -forms $\omega^{(r)} \in \Omega_{\square}^p(\mathcal{U})$ and $\omega^* \in \Omega_{\square}^p(\mathcal{U})$ so that $\omega^{(r)} = \omega^*$ for sufficiently large $r \geq 0$. By Lemma 8.8, there is a $(p-1)$ -form $D_{\mathcal{U}}^{(r)}(\omega) \in \Omega_{\square}^{p-1}(X)$ such that $dD_{\mathcal{U}}^{(r)}(\omega) = \omega^{(r+1)} - \omega^{(r)}$ if $d\omega = 0$. If we assume $\text{res}^*([\omega]) = 0$, then we may assume $\text{res}(\omega) = 0$ and $d\omega = 0$, and so we obtain $\omega^* = 0$ and $\omega = d\left\{\sum_{r=0}^N D_{\mathcal{U}}^{(r)}(\omega) - D_{\{X\}}(\omega)\right\}$ for sufficiently large $N \geq 0$, in other words, ω is an exact form and cohomologous to zero. Thus $\text{res}^* : H_{\square}^*(X) \rightarrow H_{\square}^*(\mathcal{U})$ is an monomorphism. \square

9. MAYER-VIETORIS SEQUENCE AND THEOREM OF DE RHAM

Theorem 9.1. *Let $\mathcal{U} = \{U_1, U_2\}$ be any open covering of a differentiable space X . The canonical inclusions $i_t : U_1 \cap U_2 \hookrightarrow U_t$ and $j_t : U_t \hookrightarrow X$, $t = 1, 2$, induce $\psi^{\sharp} : \Omega_{\square}^p(\mathcal{U}) \rightarrow \Omega_{\square}^p(U_1) \oplus \Omega_{\square}^p(U_2)$ and $\phi^{\sharp} : \Omega_{\square}^p(U_1) \oplus \Omega_{\square}^p(U_2) \rightarrow \Omega_{\square}^p(U_1 \cap U_2)$ by $\psi^{\sharp}(\omega) = i_1^{\sharp}\omega \oplus i_2^{\sharp}\omega$ and $\phi^{\sharp}(\eta_1 \oplus \eta_2) = j_1^{\sharp}\eta_1 - j_2^{\sharp}\eta_2$. Then we obtain the following long exact sequence.*

$$\begin{aligned} H_{\square}^0(X) \rightarrow \cdots \rightarrow H_{\square}^p(X) &\xrightarrow{\psi^*} H_{\square}^p(U_1) \oplus H_{\square}^p(U_2) \xrightarrow{\phi^*} H_{\square}^p(U_1 \cap U_2) \\ &\rightarrow H_{\square}^{p+1}(X) \xrightarrow{\psi^*} H_{\square}^{p+1}(U_1) \oplus H_{\square}^{p+1}(U_2) \xrightarrow{\phi^*} H_{\square}^{p+1}(U_1 \cap U_2) \rightarrow \cdots, \end{aligned}$$

where ψ^* and ϕ^* are induced from ψ^{\sharp} and ϕ^{\sharp} .

Proof: Since $H_{\square}^*(X) = H_{\square}^*(\mathcal{U})$ by Theorem 8.9, we are left to show long exact sequence

$$0 \longrightarrow \Omega_{\square}^p(\mathcal{U}) \xrightarrow{\psi^{\sharp}} \Omega_{\square}^p(U_1) \oplus \Omega_{\square}^p(U_2) \xrightarrow{\phi^{\sharp}} \Omega_{\square}^p(U_0) \longrightarrow 0, \quad U_0 = U_1 \cap U_2.$$

(exactness at $\Omega_{\square}^p(\mathcal{U})$): Assume $\psi^{\sharp}(\omega) = 0$, and so $j_t^{\sharp}\omega = 0$ for $t = 1, 2$. Then for any $P : \square^n \rightarrow X$, $P \in \mathcal{E}_{\square}^{\mathcal{U}}$, we have either $\text{Im } P \subset U_1$ or $\text{Im } P \subset U_2$. Therefore, we may assume either $P \in \mathcal{E}_{\square}^{U_0}$ or $P \in \mathcal{E}_{\square}^{U_1}$. In each case, we have $\omega_{\underline{n}}(P) = 0$, which implies that $\omega = 0$. Thus ψ^{\sharp} is monic.

(exactness at $\Omega_{\square}^p(U_1) \oplus \Omega_{\square}^p(U_2)$): Assume $\phi^\sharp(\eta^{(1)} \oplus \eta^{(2)}) = 0$, and so $i_1^\sharp \eta^{(1)} = i_2^\sharp \eta^{(2)}$.

Then we can construct a differential p -form $\omega \in \Omega_{\square}^p(\mathcal{U})$ as follows: for any $P \in \mathcal{E}_{\square}^{\mathcal{U}}$, we have $\text{Im } P \subset U_t$ for either $t = 1$ or 2 . Using this t , we define $\omega_{\underline{n}}(P) = \eta_{\underline{n}}^{(t)}(P)$. If $\text{Im } P \subset U_1$ and $\text{Im } P \subset U_2$, then we have $\text{Im } P \subset U_1 \cap U_2$, and hence $\eta_{\underline{n}}^{(1)}(P) = \eta_{\underline{n}}^{(2)}(P)$, since $i_1^\sharp \eta^{(1)} = i_2^\sharp \eta^{(2)}$. It implies that ω is well-defined and that $\psi^\sharp(\omega) = \eta^{(1)} \oplus \eta^{(2)}$. The converse is clear and we have $\text{Ker } \phi^\sharp = \text{Im } \psi^\sharp$.

(exactness at $\Omega_{\square}^p(U_0)$): Assume $\kappa \in \Omega_{\square}^p(U_0)$. We define $\kappa^{(t)} \in \Omega_{\square}^p(U_t)$, $t = 1, 2$ as follows: for any $P_t \in \mathcal{E}_{\square}^{U_t}$, we define $\kappa_{\underline{n}}^{(t)}(P_t)(\mathbf{x})$ by $(-1)^{t-1} \rho_{P_t}^{(3-t)}(\mathbf{x}) \cdot \kappa_{\underline{n}}(P_t)(\mathbf{x})$ if $\mathbf{x} \in P_t^{-1}(U_{3-t})$ and by 0 if $\mathbf{x} \notin \text{Supp } \rho_{P_t}^{3-t}$. Hence $\kappa^{(t)}$ is well-defined satisfying $i_1^\sharp \kappa^{(1)} - i_2^\sharp \kappa^{(2)} = \kappa$, and we obtain $\kappa = \phi^\sharp(\kappa^{(1)} \oplus \kappa^{(2)})$. Thus ϕ^\sharp is an epimorphism.

Since ψ^\sharp and ϕ^\sharp are clearly cochain maps, we obtain the desired long exact sequence. \square

Now let us turn our attention to the differential forms with compact support. Let $X = (X, \mathcal{E}^X)$ be a weakly-separated differentiable space.

Definition 9.2. Let U be an open set in X , $F \subset U$ a closed set in X and \mathcal{U} an open covering of U . We denote by $\Omega_{\square_c}^p(\mathcal{U}; F)$ the set of all $\omega \in \Omega_{\square_c}^p(\mathcal{U})$ satisfying $\text{Supp } \omega_{\underline{n}}(P) \subset P^{-1}(F)$ for any $P \in \mathcal{E}(\square^n)$. For example, any $\omega \in \Omega_{\square_c}^p(\mathcal{U})$ is in $\Omega_{\square_c}^p(\mathcal{U}; F)$ if $F \supset K_\omega$. We denote by $H_{\square_c}^*(\mathcal{U}; F)$ the cohomology of $\Omega_{\square_c}^*(\mathcal{U}; F)$ a differential subalgebra of $\Omega_{\square_c}^*(\mathcal{U})$.

Definition 9.3. Let U and V be open sets and $F \subset U$ and $G \subset V$ be closed sets in X so that $(U, F) \subset (V, G)$, and $j : (U, F) \hookrightarrow (V, G)$ be the canonical inclusion. Let \mathcal{U} and \mathcal{V} be open coverings of U and V , respectively, satisfying $F \cap W = \emptyset$ for any $W \in \mathcal{V} \setminus \mathcal{U}$. Then a homomorphism $j_\sharp : \Omega_{\square_c}^p(\mathcal{U}; F) \rightarrow \Omega_{\square_c}^p(\mathcal{V}; G)$ is defined as follows: for any $\omega \in \Omega_{\square_c}^p(\mathcal{U}; F)$, $j_\sharp \omega \in \Omega_{\square_c}^p(\mathcal{V}; G)$ is given, for $Q \in \mathcal{E}^{\mathcal{V}}(\square^m)$, by

$$\begin{cases} (j_\sharp \omega)_{\underline{m}}(Q) = \omega_{\underline{m}}(Q), & \text{if } \text{Im } Q \subset W \text{ for some } W \in \mathcal{U}, \\ (j_\sharp \omega)_{\underline{m}}(Q) = 0, & \text{if } \text{Im } Q \subset W \text{ for some } W \in \mathcal{V} \setminus \mathcal{U} \end{cases}$$

with $K_{j_\sharp \omega} = K_\omega \subset F \subset G$. In particular, for any $\omega \in \Omega_{\square_c}^p(\mathcal{U})$, we have $\omega \in \Omega_{\square_c}^p(\mathcal{U}; K_\omega)$, and so we obtain $j_\sharp \omega \in \Omega_{\square_c}^p(j_\sharp \mathcal{U}_\omega; K_\omega) \subset \Omega_{\square_c}^p(j_\sharp \mathcal{U}_\omega)$, $j_\sharp \mathcal{U}_\omega = \mathcal{U} \cup \{V \setminus K_\omega\}$.

Remark 9.4. In Definition 9.3, the map j_\sharp induced from $j : (U, F) \hookrightarrow (V, G)$ satisfies that $(j_\sharp \omega)_{\underline{m}}(j \circ Q) = \omega_{\underline{m}}(Q)$ for any $m \geq 0$ and $Q \in \mathcal{E}^{\mathcal{U}}(\square^m)$.

Proposition 9.5. Let $X = (X, \mathcal{E}^X)$ be a weakly-separated differentiable space and U and V open in X . Then the correspondence $\Omega_{\square_c}^*(U) \ni \omega \mapsto j_\sharp \omega \in \Omega_{\square_c}^*(j_\sharp \mathcal{U}_\omega)$ induced from the canonical inclusion $j : U \hookrightarrow V$ induces a homomorphism $j_* : H_{\square_c}^*(U) \rightarrow H_{\square_c}^*(V)$, since there is a canonical isomorphism $H_{\square_c}^*(j_\sharp \mathcal{U}_\omega) \cong H_{\square_c}^*(V)$ by Theorem 8.9.

Proof: Let $\omega, \eta \in \Omega_{\square_c}^*(U)$. Then $K = K_\omega \cup K_\eta$ is compact in U and hence in X . Let $\mathcal{U} = \{U, V \setminus K\}$, which is a finer open covering of \mathcal{U}_ω and \mathcal{U}_η , and hence both isomorphisms $H_{\square_c}^*(V) \rightarrow H_{\square_c}^*(\mathcal{U}_\omega)$ and $H_{\square_c}^*(V) \rightarrow H_{\square_c}^*(\mathcal{U}_\eta)$ defined in Theorem 8.9 go through the isomorphism $H_{\square_c}^*(V) \rightarrow H_{\square_c}^*(\mathcal{U})$. Thus the homomorphisms $H_{\square_c}^*(\mathcal{U}) \rightarrow H_{\square_c}^*(\mathcal{U}_\omega)$ and $H_{\square_c}^*(\mathcal{U}) \rightarrow H_{\square_c}^*(\mathcal{U}_\eta)$ are also isomorphisms. By definition, $j_\#(\omega + \eta) = j_\#(\omega) + j_\#(\eta)$ in $\Omega_{\square_c}^*(\mathcal{U})$, and hence $j_*([\omega + \eta]) = j_*([\omega]) + j_*([\eta])$ in $H_{\square_c}^*(X)$ for any $[\omega], [\eta] \in H_{\square_c}^*(U)$. \square

Theorem 9.6. *Let $\mathcal{U} = \{U_1, U_2\}$ be an open covering of a weakly-separated differentiable space X with a normal partition of unity $\{\rho^{(1)}, \rho^{(2)}\}$ belonging to \mathcal{U} , i.e., there are closed subsets $\{G_1, G_2\}$ such that $G_t \subset U_t$ and $\text{Supp } \rho_{\underline{n}}^{(t)}(P) \subset P^{-1}(G_t)$ for any $P \in \mathcal{E}(\square^n)$, $t = 1, 2$. Then we have $G_1 \cup G_2 = X$. Let $G_0 = G_1 \cap G_2 \subset U_0 = U_1 \cap U_2$. The canonical inclusions $i_t : U_1 \cap U_2 \hookrightarrow U_t$ and $j_t : U_t \hookrightarrow X$, $t = 1, 2$, induce $\phi_* : H_{\square_c}^p(U_0) \rightarrow H_{\square_c}^p(U_1) \oplus H_{\square_c}^p(U_2)$ and $\psi_* : H_{\square_c}^p(U_1) \oplus H_{\square_c}^p(U_2) \rightarrow H_{\square_c}^p(X)$ by $\phi_*([\omega]) = i_{1*}[\omega] \oplus i_{2*}[\omega]$ and $\psi_*([j_1] \oplus [j_2]) = j_{1*}[j_1] - j_{2*}[j_2]$. Then we obtain the following long exact sequence.*

$$\begin{aligned} H_{\square_c}^0(U_0) \rightarrow \cdots \rightarrow H_{\square_c}^p(U_0) \xrightarrow{\phi_*} H_{\square_c}^p(U_1) \oplus H_{\square_c}^p(U_2) \xrightarrow{\psi_*} H_{\square_c}^p(X) \\ \xrightarrow{d_*} H_{\square_c}^{p+1}(U_0) \xrightarrow{\phi_*} H_{\square_c}^{p+1}(U_1) \oplus H_{\square_c}^{p+1}(U_2) \xrightarrow{\psi_*} H_{\square_c}^{p+1}(X) \rightarrow \cdots \end{aligned}$$

Proof: For any closed subsets $G'_t \supset G_t$ in U_t , there is a following short exact sequence.

$$0 \longrightarrow \Omega_{\square_c}^p(\mathcal{U}_0; G'_0) \xrightarrow{\phi_{\natural}} \Omega_{\square_c}^p(\mathcal{U}_1; G'_1) \oplus \Omega_{\square_c}^p(\mathcal{U}_2; G'_2) \xrightarrow{\psi_{\natural}} \Omega_{\square_c}^p(\mathcal{U}_3; X) \longrightarrow 0,$$

where $G'_0 = G'_1 \cap G'_2$, $\mathcal{U}_0 = \{U_0\}$, $\mathcal{U}_t = \{U_0, U_t \setminus G'_{3-t}\}$, $t = 1, 2$ and $\mathcal{U}_3 = \{U_0, U_1 \setminus G'_2, U_2 \setminus G'_1\}$, which are open coverings of U_0 , U_t and X , respectively.

(exactness at $\Omega_{\square_c}^p(\mathcal{U}_0; G'_0)$): Assume $\phi_{\natural}(\omega) = 0$. Then $i_{1\#}(\omega) = i_{2\#}(\omega) = 0$. Since $i_{1\#}(\omega)$ is an extension of ω , we obtain $\omega = 0$. Thus ϕ_{\natural} is a monomorphism.

(exactness at $\Omega_{\square_c}^p(\mathcal{U}_1; G'_1) \oplus \Omega_{\square_c}^p(\mathcal{U}_2; G'_2)$): Assume $\psi_{\natural}(\eta^{(1)} \oplus \eta^{(2)}) = 0$. Then we have $j_{1\#}(\eta^{(1)}) = j_{2\#}(\eta^{(2)})$. For any plot $P : \square^n \rightarrow X$, we obtain $j_{1\#}(\eta^{(1)})_{\underline{n}}(P) = j_{2\#}(\eta^{(2)})_{\underline{n}}(P)$. So, for any plot $Q : \square^m \rightarrow U_0$, $\eta_B^{(1)}(i_1 \circ Q) = j_1^\# \eta_{\underline{m}}^{(1)}(j_1 \circ i_1 \circ Q) = j_2^\# \eta_{\underline{m}}^{(2)}(j_2 \circ i_2 \circ Q) = \eta_{\underline{m}}^{(2)}(i_2 \circ Q)$. Then, we define $\eta^{(0)} \in \Omega_{\square_c}^p(U_0)$ by $\eta_{\underline{m}}^{(0)}(Q) = \eta_{\underline{m}}^{(1)}(i_1 \circ Q) = \eta_{\underline{m}}^{(2)}(i_2 \circ Q)$. On the other hand, $K_{j_t \eta^{(t)}} = K_{\eta^{(t)}}$ by definition, and hence we obtain

$$\text{Supp } \eta_{\underline{m}}^{(0)}(Q) = \text{Supp } \eta_{\underline{m}}^{(1)}(i_1 \circ Q) = \text{Supp } \eta_{\underline{m}}^{(2)}(i_2 \circ Q) \subset Q^{-1}(K_{\eta^{(1)}} \cap K_{\eta^{(2)}}).$$

Then we have $\eta^{(0)} \in \Omega_{\square_c}^p(\mathcal{U}_0)$, for $K_{\eta^{(0)}} = K_{\eta^{(1)}} \cap K_{\eta^{(2)}}$ is compact in U_0 , which satisfies $\phi_{\natural}(\eta^{(0)}) = (\eta^{(1)}, \eta^{(2)})$. Thus $(\eta^{(1)}, \eta^{(2)})$ is in the image of ϕ_{\natural} . The other direction is clear by definition and it implies the exactness at $\Omega_{\square_c}^p(\mathcal{U}_1; G'_1) \oplus \Omega_{\square_c}^p(\mathcal{U}_2; G'_2)$.

(exactness at $\Omega_{\square_c}^p(\mathcal{U}_3; X)$): Assume $\kappa \in \Omega_{\square_c}^p(\mathcal{U}_3; X)$. For any plot $P_t : \square^{n_t} \rightarrow U_t$, we define $\kappa_{\underline{n_t}}^{(t)}(P_t)(\mathbf{x})$ by $(-1)^{t-1} \rho_{\underline{n_t}}^{(t)}(P_t)(\mathbf{x}) \cdot \kappa_{\underline{n_t}}(j_t \circ P_t)(\mathbf{x})$ if $\mathbf{x} \in P_t^{-1}(U_0)$ and by 0

if $\mathbf{x} \notin \text{Supp } \rho_{n_t}^{(t)}(P_t)$. Then $\kappa^{(t)}$ is a differential p -form on U_t and $\kappa^{(t)} \in \Omega_{\square_c}^p(U_t)$ for $K_{\kappa^{(t)}} = K_\kappa \cap G_t \subset G'_t$ is compact in U_t . Then we have $\psi_{\natural}(\kappa^{(1)} \oplus \kappa^{(2)}) = \kappa$, and hence κ is in the image of ψ_{\natural} . Thus ψ_{\natural} is an epimorphism.

Since ϕ_{\natural} and ψ_{\natural} are clearly cochain maps, we obtain the following long exact sequence.

$$\begin{aligned} H_{\square_c}^0(\mathcal{U}_0; G'_0) \rightarrow \cdots \rightarrow H_{\square_c}^p(\mathcal{U}_0; G'_0) \xrightarrow{\bar{\phi}_*} H_{\square_c}^p(\mathcal{U}_1; G'_1) \oplus H_{\square_c}^p(\mathcal{U}_2; G'_2) \xrightarrow{\bar{\psi}_*} H_{\square_c}^p(\mathcal{U}_3) \\ \xrightarrow{\bar{d}_*} H_{\square_c}^{p+1}(\mathcal{U}_0; G'_0) \xrightarrow{\bar{\phi}_*} H_{\square_c}^{p+1}(\mathcal{U}_1; G'_1) \oplus H_{\square_c}^{p+1}(\mathcal{U}_2; G'_2) \xrightarrow{\bar{\psi}_*} H_{\square_c}^{p+1}(\mathcal{U}_3) \rightarrow \cdots \end{aligned}$$

So we can define connecting homomorphism $d_* : H_{\square_c}^p(X) \xrightarrow{\text{res}^*} H_{\square_c}^p(\mathcal{U}_3) \xrightarrow{\bar{d}_*} H_{\square_c}^{p+1}(\mathcal{U}_0; G'_0) \rightarrow H_{\square_c}^{p+1}(U_0)$ where the latter map is induced from the natural inclusion $\Omega_{\square_c}^{p+1}(\mathcal{U}_0; G'_0) \subset \Omega_{\square_c}^{p+1}(U_0) = \Omega_{\square_c}^{p+1}(U_0)$, which fits in with the following commutative ladder.

$$\begin{array}{ccccccc} H_{\square_c}^p(\mathcal{U}_0; G'_0) & \xrightarrow{\bar{\phi}_*} & H_{\square_c}^p(\mathcal{U}_1; G'_1) \oplus H_{\square_c}^p(\mathcal{U}_2; G'_2) & \xrightarrow{\bar{\psi}_*} & H_{\square_c}^p(\mathcal{U}_3) & \xrightarrow{\bar{d}_*} & H_{\square_c}^{p+1}(\mathcal{U}_0; G'_0) \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ H_{\square_c}^p(\mathcal{U}_0) & & H_{\square_c}^p(\mathcal{U}_1) \oplus H_{\square_c}^p(\mathcal{U}_2) & & H_{\square_c}^p(\mathcal{U}_3) & & H_{\square_c}^{p+1}(\mathcal{U}_0) \\ \parallel & & \cong \uparrow \text{res}^* \oplus \text{res}^* & & \cong \uparrow \text{res}^* & & \parallel \\ H_{\square_c}^p(U_0) & \xrightarrow{\phi_*} & H_{\square_c}^p(U_1) \oplus H_{\square_c}^p(U_2) & \xrightarrow{\psi_*} & H_{\square_c}^p(X) & \xrightarrow{d_*} & H_{\square_c}^{p+1}(U_0) \end{array}$$

Using these diagrams, we show the desired exactness as follows.

(exactness at $H_{\square_c}^p(U_0)$): Assume $\phi_*([\omega]) = 0$. Let $G'_t = G_t \cup K_\omega$, $t = 0, 1, 2$. Then $[\omega] \in H_{\square_c}^p(\mathcal{U}_0; G'_0)$ satisfying $\bar{\phi}_*([\omega])$ is zero in $H_{\square_c}^p(\mathcal{U}_1) \oplus H_{\square_c}^p(\mathcal{U}_2)$. Hence there is $\sigma^{(1)} \oplus \sigma^{(2)} \in \Omega_{\square_c}^p(\mathcal{U}_1) \oplus \Omega_{\square_c}^p(\mathcal{U}_2)$ such that $d\sigma^{(1)} \oplus d\sigma^{(2)} = \phi_{\natural}(\omega)$. Then we may expand G'_t as $G'_t = G_t \cup K_\omega \cup K_{\sigma^{(t)}}$, $t = 1, 2$ and $G'_0 = G'_1 \cap G'_2$, so that we obtain $\bar{\phi}_*([\omega]) = 0$, and hence $[\omega] \in \text{Im } \bar{d}_*$ in $H_{\square_c}^{p+1}(\mathcal{U}_0; G'_0)$. Thus $[\omega]$ is in the image of d_* .

(exactness at $H_{\square_c}^p(U_1) \oplus H_{\square_c}^p(U_2)$): Assume $\psi_*([\eta^{(1)}] \oplus [\eta^{(2)}]) = 0$. Let $G'_t = G_t \cup K_{\eta^{(t)}}$, $t = 1, 2$ and $G'_0 = G'_1 \cap G'_2$, so that $[\eta^{(1)}] \oplus [\eta^{(2)}] \in H_{\square_c}^p(\mathcal{U}_1; G'_1) \oplus H_{\square_c}^p(\mathcal{U}_2; G'_2)$ and $\bar{\psi}_*([\eta^{(1)}] \oplus [\eta^{(2)}]) = 0$ in $H_{\square_c}^p(\mathcal{U}_3) \cong H_{\square_c}^p(X)$. Then we obtain $[\eta^{(1)}] \oplus [\eta^{(2)}] \in \text{Im } \bar{\phi}_*$ in $H_{\square_c}^p(\mathcal{U}_1; G'_1) \oplus H_{\square_c}^p(\mathcal{U}_2; G'_2)$, and hence $[\eta^{(1)}] \oplus [\eta^{(2)}]$ is in the image of ϕ_* .

(exactness at $H_{\square_c}^p(X)$): Assume $d_*([\kappa]) = 0$. Then there is $\sigma \in \Omega_{\square_c}^p(U_0)$ such that $d_{\natural}(\kappa) = d\sigma$ in $\Omega_{\square_c}^{p+1}(U_0)$. Let $G'_t = G_t \cup K_\sigma$, $t = 0, 1, 2$. Then we may assume $\sigma \in \Omega_{\square_c}^p(\mathcal{U}_0; G'_0)$ satisfying $d_{\natural}(\kappa) = d\sigma$ in $\Omega_{\square_c}^{p+1}(\mathcal{U}_0; G'_0)$, and hence $[\kappa] \in \text{Im } \bar{\psi}_*$ in $H_{\square_c}^p(\mathcal{U}_3)$. Thus $[\kappa]$ is in the image of ψ_* .

The other directions are clear by definition, and it completes the proof of the theorem. \square

Let **Topology** be the category of topological spaces and continuous maps. Then there are natural embeddings $\mathbf{Topology} \hookrightarrow \mathbf{Differentiable}$ and $\mathbf{Topology} \hookrightarrow \mathbf{Diffeology}$.

Let $X = (X, \{X^{(n)}; n \geq -1\})$ be a topological CW complex embedded in the category **Diffeology** or **Differentiable** with the set of n -balls $\{B_j^n\}$ indexed by $j \in J_n$. Then we have open sets $U = X^{(n)} \setminus X^{(n-1)}$ and $V = X^{(n)} \setminus (\bigcup_{j \in J_n} \{\mathbf{0}_j\})$ in $X^{(n)}$, where $\mathbf{0}_j \in B_j^n$ denotes the element corresponding to $\mathbf{0} \in B^n = \{\mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| \leq 1\}$ the origin of \mathbb{R}^n .

A ball $B_j^n = B^n$ (if we disregard the indexing) has a nice open covering given by $\{\text{Int } B_j^n, B_j^n \setminus \{\mathbf{0}\}\}$ with a partition of unity $\{\rho_1^{(j)}, \rho_2^{(j)}\}$ as follows: $\rho_1^{(j)} = 1 - \rho_2^{(j)}$ and $\rho_2^{(j)}(\mathbf{x}) = \lambda(\|\mathbf{x}\|)$ for small $a > 0$. Thus $\mathcal{U} = \{U, V\}$ is a nice open covering of $X^{(n)}$ with a normal partition of unity $\{\rho^U, \rho^V\}$ in which ρ^U is a zero-extension of $\rho_1^{(j)}$'s on the union of balls and $\rho^V = 1 - \rho^U$. Then U is smoothly homotopy equivalent to discrete points each of which is $\mathbf{0}_j \in B_j^n$ for some $j \in J_n$ and V is smoothly homotopy equivalent to $X^{(n-1)}$. By comparing Mayer-Vietoris sequences associated to \mathcal{U} in Theorem 2.3 with that in Theorem 9.1 for $X = X^{(n)}$, we obtain the following result using Remark 1.18 together with so-called five lemma, by using standard homological methods inductively on n .

Theorem 9.7. *For a CW complex X , there are natural isomorphisms*

$$H_{\mathcal{D}}^q(X) \cong H_{\mathcal{C}}^q(X) \cong H_{\square}^q(X) \cong H^q(X, \mathbb{R}) \cong \text{Hom}(H_q(X), \mathbb{R}),$$

for any $q \geq 0$, and hence we have $H_{\mathcal{D}}^1(X) \cong H_{\mathcal{C}}^1(X) \cong H_{\square}^1(X) \xrightarrow{\rho} \text{Hom}(\pi_1(X), \mathbb{R})$.

Conjecture 9.8. *For a CW complex X , there are natural isomorphisms*

$$H_{\mathcal{D}_c}^q(X) \cong H_{\mathcal{C}_c}^q(X) \cong H_{\square_c}^q(X), \text{ for any } q \geq 0.$$

It would be possible to determine $H_{\square}^*(X)$ and $H_{\square_c}^*(X)$ by using standard methods in algebraic topology even if X is not a topological CW complex, while we do not know how to determine $H_{\mathcal{D}}^*(X)$, $H_{\mathcal{C}}^*(X)$, $H_{\mathcal{D}_c}^*(X)$ nor $H_{\mathcal{C}_c}^*(X)$, if we do not find out any appropriate nice open covering (with a normal partition of unity) on X .

APPENDIX A. SMOOTH CW COMPLEX

A smooth CW complex $X = (X, \{X^{(n)}; n \geq -1\})$ is a differentiable or diffeological space built up from $X^{(-1)} = \emptyset$ by inductively attaching n -balls $\{B_j^n; j \in J_n\}$ by C^∞ maps from their boundary spheres $\{S_j^{n-1}; j \in J_n\}$ to $n-1$ -skeleton $X^{(n-1)}$ to obtain n -skeleton $X^{(n)}$, $n \geq 0$, where the differentiable structures of balls and boundary spheres are given by their manifold structures. Thus a plot in $X^{(n)}$ is a map $P : A \rightarrow X$ with an open covering $\{A_\alpha; \alpha \in \Lambda\}$ of A such that $P(A_\alpha)$ is in $X^{(n-1)}$ or B_j^n for some $j \in J_n$ and $P|_{A_\alpha}$ is a plot of $X^{(n-1)}$ or B_j^n , respectively. Then X is the colimit of $\{X^{(n)}\}$ in **Differentiable** or **Diffeology**.

Let $X = (X, \{X^{(n)}\})$ be a smooth CW complex in either Differentiable or Diffeology with the set of n -balls $\{B_j^n; j \in J_n\}$. Then for any plot $P : A \rightarrow X^{(n)}$, there is an open covering $\{A_\alpha\}$ of A , such that $P(A_\alpha)$ is in either $X^{(n-1)}$ or B_j^n for some $j \in J_n$ and $P_\alpha = P|_{A_\alpha}$ is a plot of $X^{(n-1)}$ or B_j^n , respectively. Let $U = X^{(n)} \setminus X^{(n-1)}$ and $V = X^{(n)} \setminus (\bigcup_{j \in J_n} \{\mathbf{0}_j\})$, where $\mathbf{0}_j \in B_j^n$ denotes the element corresponding to $\mathbf{0} \in B^n$.

Case $(\text{Im } P_\alpha \subset X^{(n-1)})$: $P_\alpha^{-1}(U) = \emptyset$, and $P_\alpha^{-1}(V) = A_\alpha$.

Case $(\text{Im } P_\alpha \subset B_j^n)$: $P_\alpha^{-1}(U) = P_\alpha^{-1}(\text{Int } B_j^n)$, and $P_\alpha^{-1}(V) = P_\alpha^{-1}(B_j^n \setminus \{\mathbf{0}_j\})$.

In each case, $P_\alpha^{-1}(U)$ and $P_\alpha^{-1}(V)$ are open in A_α and hence in A , which implies that $P^{-1}(U)$ and $P^{-1}(V)$ are open in A for any plot P . Thus U and V are open sets in $X^{(n)}$.

Similarly to the case when X is a topological CW complex, $\mathcal{U} = \{U, V\}$ is a nice open covering of $X^{(n)}$ with a normal partition of unity $\{\rho^U, \rho^V\}$, since λ is a smooth function. Then, similar arguments for a topological CW complex lead us to the following result.

Theorem A.1. *For a smooth CW complex X , there are natural isomorphisms*

$$H_{\mathcal{D}}^q(X) \cong H_{\mathcal{C}}^q(X) \cong H_{\underline{\square}}^q(X) \cong H^q(X, \mathbb{R}) \cong \text{Hom}(H_q(X), \mathbb{R}),$$

for any $q \geq 0$, and hence we have $H_{\mathcal{D}}^1(X) \cong H_{\mathcal{C}}^1(X) \cong H_{\underline{\square}}^1(X) \xrightarrow{\rho} \text{Hom}(\pi_1(X), \mathbb{R})$.

Conjecture A.2. *For a smooth CW complex X , there are natural isomorphisms*

$$H_{\mathcal{D}_c}^q(X) \cong H_{\mathcal{C}_c}^q(X) \cong H_{\underline{\square}_c}^q(X), \text{ for any } q \geq 0.$$

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